

# Arguments for a single additive constant for potentials

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While the arguments put forward in Domsta’s article look mathematically sound – denying, for example, the existence of the Laplacian of a potential where it is not differentiable – their scope is too narrow. With the restriction of potentials to the class of (twice) differentiable functions, we would not even be able to define the potential of a point charge.

A standard way to derive the Poisson equation of electrodynamics is to start from Coulomb’s law for a point charge and to show that the surface integral of its electric field over any surface enclosing the charge will give  $4\pi$  times the charge (in the cgs system), producing a  $\delta$  function divergence of the field. Moreover, the curl of the field is zero. This allows to define the field as the negative gradient of a potential satisfying the Poisson equation with a  $\delta$  function source. The potential goes as  $1/r$  and if its singularity at the origin, rendering it non-differentiable there, were excluded from consideration, we would not have any useful equation to work with. Since the Poisson equation for charge densities is obtained via generalization of the point charge case, it would be not so straightforward to arrive at it, if we didn’t allow it for point charges to begin with.

The way out is standard for physicists. Instead of functions, consider more general objects, i.e. distributions. These are always differentiable by definition.<sup>1</sup>

In the light of this, I would like to reconsider some of the examples given in Domsta’s article. First, note that a potential  $V(x)$  going to infinity as  $x \rightarrow \infty$  is not a problem. The harmonic oscillator is *the* paradigm of quantum mechanics and its potential is not bounded from above. This is not completely artificial: a potential that has similar properties is the confinement potential for quarks in elementary particle physics. It also increases on separation of differently colored quarks up to the point where the potential energy is sufficient to create one or several new quarks allowing the appearance of “white” (i.e. uncolored) elementary particles.

Consider now the potential of a bounded spherical mass distribution, homogeneous on its support. I write it as follows

$$\Phi_{\text{grav}}(\mathbf{r}) = \left( \frac{1}{2}GM \frac{r^2}{R^3} + C_1 \right) \Theta(R - r) + \left( C_2 - \frac{GM}{r} \right) \Theta(r - R). \quad (1)$$

Herein,  $\Theta(x)$  is the Heaviside function (distribution), defined by

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}, \quad \Rightarrow \quad \Theta'(x) = \delta(x). \quad (2)$$

It is possible to assign a value to  $\Theta(0)$ , the most plausible one being  $1/2$  [requiring  $\Theta(x) + \Theta(-x) = 1$ ], but the important property here is that the  $\Theta$  function is the integral of the  $\delta$  function(al).

First, it should be clear that the definition (1) is equivalent to the one by Domsta, if we set  $C_1 = C$  and  $C_2 = C + \frac{3}{2}GM \frac{1}{R}$ . I will, however, keep the constants general at this point, because I want to show two things: a) the potential can be made to satisfy the appropriate Poisson

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<sup>1</sup>For a physicist, this may collapse to the simple notion of considering all functions as twice differentiable. The mathematician will rigorously prove that distributions are always differentiable and that this follows from their definition either via sequences of functions or via functionals. This includes distributions such as potentials that are defined piecewise and have jump discontinuities or discontinuous slopes.

equation also at  $r = R$  and hence define the field correctly there and b) this is true providing we choose the difference  $C_2 - C_1$  correctly. So only one additive constant is arbitrary, not two of them.

To proceed, we calculate the Laplacian of the potential by direct differentiation. In spherical coordinates, the radial part of the Laplacian is given by  $\Delta_r = \frac{1}{r} \frac{\partial^2}{\partial r^2} r$  and its angular part does not contribute anything when applied to a distribution of  $r$ . We find

$$\begin{aligned}
\Delta \Phi_{\text{grav}}(\mathbf{r}) &= \Delta_r \Phi_{\text{grav}}(\mathbf{r}) \\
&= \frac{1}{r} \frac{\partial^2}{\partial r^2} r \left[ \left( \frac{1}{2} GM \frac{r^2}{R^3} + C_1 \right) \Theta(R-r) + \left( C_2 - \frac{GM}{r} \right) \Theta(r-R) \right] \\
&= \frac{1}{r} \frac{\partial^2}{\partial r^2} \left[ \left( \frac{1}{2} GM \frac{r^3}{R^3} + C_1 r \right) \Theta(R-r) + (C_2 r - GM) \Theta(r-R) \right] \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( \frac{3}{2} GM \frac{r^2}{R^3} + C_1 \right) \Theta(R-r) + C_2 \Theta(r-R) \right. \\
&\quad \left. + \underbrace{\left( (C_2 - C_1)r - GM - \frac{1}{2} GM \frac{r^3}{R^3} \right)}_{((C_2 - C_1)R - GM - \frac{1}{2} GM) \delta(r-R)} \delta(r-R) \right] \\
&= \frac{1}{r} \left[ 3GM \frac{r}{R^3} \Theta(R-r) + \underbrace{\left( C_2 - C_1 - \frac{3}{2} GM \frac{r^2}{R^3} \right)}_{(C_2 - C_1 - \frac{3}{2} GM \frac{1}{R}) \delta(r-R)} \delta(r-R) \right. \\
&\quad \left. + \left( (C_2 - C_1)R - \frac{3}{2} GM \right) \delta'(r-R) \right] \\
&= \frac{3GM}{R^3} \Theta(R-r) + \frac{1}{r} \left( C_2 - C_1 - \frac{3}{2} GM \frac{1}{R} \right) \delta(r-R) \\
&\quad + \frac{1}{r} \left( (C_2 - C_1)R - \frac{3}{2} GM \right) \delta'(r-R). \tag{3}
\end{aligned}$$

We have used the symmetry of the  $\delta$  function in this derivation, i.e.  $\frac{\partial}{\partial r} \Theta(R-r) = -\delta(R-r) = -\delta(r-R)$ .

Now consider the mass density described by the last line of (3). The derivative of the  $\delta$  function describes the density of a layer of (mathematical) mass dipoles. We do not want such a layer to be present. We can avoid it by setting

$$C_2 = C_1 + \frac{3}{2} \frac{GM}{R}. \tag{4}$$

But then the prefactor of the  $\delta$  function also becomes zero, so we have no layer of point masses at  $r = R$  either and end up with

$$\Delta \Phi_{\text{grav}}(\mathbf{r}) = \frac{3GM}{R^3} \Theta(R-r) = 4\pi \frac{GM}{4\pi R^3/3} \Theta(R-r) \stackrel{!}{=} 4\pi G \rho(r). \tag{5}$$

This defines the mass density everywhere. It is constant for  $r < R$ , zero for  $r > R$  and depends on the convention taken for the value of the Heaviside function at zero argument. Usually, this will either be the same constant as in the interior of the sphere or else half its value. Since the total mass in the surface is zero, it is clear that this ambiguity does not affect the potential.

Clearly, we can add an arbitrary constant to the potential without affecting the physics but not two arbitrary constants at  $r < R$  and at  $r > R$ . Note that the correct choice of  $C_2 - C_1$  is the

one that makes the potential continuous at  $r = R$ . (Otherwise, we would have a surface dipole layer.)

Next consider the potential of the metallic ball. Since here we have surface charges, the Poisson equation must produce a surface  $\delta$  function. The idea that it gives zero everywhere away from the surface and no result on the surface would call its usefulness into question (as well as its derivation based on point charges which represent even more singular charge densities than surface charges).

The potential reads

$$\Phi_{\text{electr}}(\mathbf{r}) = C_1 \Theta(R - r) + \left( \frac{Q}{r} + C_2 \right) \Theta(r - R) \quad (6)$$

and we should obtain ( $4\pi$  times) the (negative) charge density on taking the Laplacian:

$$\begin{aligned} \Delta \Phi_{\text{electr}}(\mathbf{r}) &= \Delta_r \Phi_{\text{electr}}(\mathbf{r}) = \frac{1}{r} \frac{\partial^2}{\partial r^2} [C_1 r \Theta(R - r) + (Q + C_2 r) \Theta(r - R)] \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left[ C_1 \Theta(R - r) + C_2 \Theta(r - R) + \underbrace{(Q + (C_2 - C_1)r) \delta(r - R)}_{(Q + (C_2 - C_1)R) \delta(r - R)} \right] \\ &= \frac{1}{r} \left[ (C_2 - C_1) \delta(r - R) + (Q + (C_2 - C_1)R) \delta'(r - R) \right]. \end{aligned} \quad (7)$$

Again, we require the prefactor of the derivative of the  $\delta$  function to be zero, because it would correspond to a layer of charge dipoles. This implies

$$C_2 - C_1 = -\frac{Q}{R} \quad (8)$$

and we have

$$\Delta \Phi_{\text{electr}}(\mathbf{r}) = -\frac{Q}{R^2} \delta(r - R) \stackrel{!}{=} -4\pi \rho(r) \quad \Rightarrow \quad \rho(r) = \frac{Q}{4\pi R^2} \delta(r - R). \quad (9)$$

That this is the appropriate surface  $\delta$  function can be seen by integrating over an infinitesimally thick layer about the surface:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{R-\varepsilon}^{R+\varepsilon} r^2 dr \int_0^\pi \sin \vartheta d\vartheta \int_0^{2\pi} d\varphi \rho(r) &= \lim_{\varepsilon \rightarrow 0^+} \int_{R-\varepsilon}^{R+\varepsilon} 4\pi \rho(r) r^2 dr \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{R-\varepsilon}^{R+\varepsilon} \frac{Q}{R^2} \underbrace{\delta(r - R) r^2}_{\delta(r - R) R^2} dr = Q. \end{aligned} \quad (10)$$

Hence, interpreting potentials and charge densities as distributions rather than functions, we may claim that the Poisson equation holds everywhere including at the surface.