The principle of stationary action

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Another small lecture seems to be appropriate. Apparently, the idea is prevalent that the laws of mechanics are expressible via a principle of least action, aka Hamilton's principle. My statements to the effect that the principle is actually one asserting only stationarity of the action, which might as well have a saddle point or a maximum for the true trajectory, were met with disbelief and attempts to invoke stability as a feature necessarily leading to a minimum of the action being realized by the physical trajectory. The best way to show that this is wrong seems to me to discuss an extremely simple counterexample, which in fact is just the harmonic oscillator. Moreover, the idea that whenever we have a system where kinetic energy T and potential energy V are defined, the Lagrangian L appearing in the definition of the action is given by $L = T - V$ will be shown to be wrong, too, again by a simple example.

Hamilton's principle at work with the harmonic oscillator

Hamilton's principle says that for a mechanical[1] system fully described by a set of generalized coordinates $\{q_i,\ i = 1 \ldots N\}$, the dynamics of the motion is such that the action given by

$$S(t_i, t_f) = \int_{t_i}^{t_f} L\left(\{\dot{q}_i\}, \{q_i\}, t\right) \mathrm{d}t\,, \tag{1}$$

where L is the Lagrangian of the system, has a stationary point with respect to small variations about the true trajectory, i.e., the dynamical path (in configuration space) actually taken by the system. The admissible variations are taken at fixed time which means that the variations of the generalized velocities are the time derivatives of the variations of the generalized coordinates. Moreover, the initial coordinates $q_i(t_i)$ and the final ones $q_i(t_f)$ are the same for the varied trajectories as for the physical one, i.e. the initial and final points are fixed and not subject to variation.

Stationarity of the action, i.e., $\delta S = 0$, required for sufficiently small variations about the true trajectory with the described boundary conditions, leads to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \qquad i = 1 \ldots N\,. \tag{2}$$

Note that no requirement about the second variation of the action has been made, and hence the vanishing of the variational derivative of S can be due to the functional S having either a minimum, a saddle point or a maximum. To see which may be the case, let us check an example.

Consider a one-dimensional harmonic oscillator. Its Lagrangian is given by

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 \tag{3}$$

i.e., we have a single coordinate $x(t)$ describing the instantaneous amplitude of the oscillator. The Euler-Lagrange equation from the principle of stationary action reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0\,, \tag{4}$$

[1] The principle was of course extended beyond mechanical systems to electrodynamics and essentially all other fields of physics.

which is the correct equation of motion of a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \ , \tag{5}$$

implying $k = m\omega^2$. Let us take as initial time $t_i = 0$ and as initial condition $x(t_i) = 0$. For the final time, we choose $t_f = \left(n + \frac{1}{8}\right) T$, where n is a non-negative integer,[2] and take as amplitude $x(t_f) = X$. The solution of the equation of motion with these boundary conditions is easily determined. Its general solution is $x(t) = A \sin \omega t + B \cos \omega t$. Then $x(0) = 0$ implies $B = 0$, and the second boundary condition leads to

$$A = \frac{X}{\sin \omega t_f} = \sqrt{2} X \ , \tag{6}$$

because $\sin \left[\omega \left(n + \frac{1}{8}\right) T\right] = \sin \left[\frac{2\pi}{T} \left(n + \frac{1}{8}\right) T\right] = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. So the physical trajectory for the boundary value problem is given by

$$x(t) = \sqrt{2} X \sin \omega t \ , \tag{7}$$

implying

$$\dot{x}(t) = \sqrt{2} X \, \omega \cos \omega t \ , \tag{8}$$

and we may easily calculate the Lagrangian

$$L_0 = mX^2\omega^2 \cos^2 \omega t - kX^2 \sin^2 \omega t = mX^2\omega^2 \cos 2\omega t \tag{9}$$

as well as the action

$$\begin{aligned} S_0(t_t, t_f) &= \int_{t_i}^{t_f} mX^2\omega^2 \cos 2\omega t \mathrm{d}t = \frac{m}{2} X^2 \, \omega \sin 2\omega t \Big|_{t_i}^{t_f} = \frac{m}{2} X^2 \, \omega \sin \left[\frac{4\pi}{T} \left(n + \frac{1}{8}\right) T\right] \\ &= \frac{m}{2} X^2 \, \omega \sin \frac{\pi}{2} = \frac{\pi}{T} m X^2 \ , \end{aligned} \tag{10}$$

which happens to be independent of n. Is this a minimum? Let us compare it with the action along a non-physical trajectory between the same initial and final points. Such a trajectory is given, for example, by

$$x(t) = X \frac{t}{t_f} \quad \Rightarrow \quad \dot{x}(t) = \frac{X}{t_f} \ , \tag{11}$$

so the Lagrangian becomes

$$L_1 = \frac{m}{2} \left(\frac{X}{t_f}\right)^2 - \frac{m}{2} \omega^2 X^2 \left(\frac{t}{t_f}\right)^2 = \frac{m}{2} \left(\frac{X}{t_f}\right)^2 \left(1 - \omega^2 t^2\right) \ . \tag{12}$$

This is easily integrated to give

$$S_1(0, t_f) = \frac{m}{2} \left(\frac{X}{t_f}\right)^2 \int_0^{t_f} \left(1 - \omega^2 t^2\right) \mathrm{d}t = \frac{m}{2} \frac{X^2}{t_f} \left(1 - \frac{1}{3} \omega^2 t_f^2\right) \ , \tag{13}$$

and it is obvious that for sufficiently large t_f, this will become smaller than $S_0(0, t_f)$. We have

$$n = 0 : \quad t_f = \frac{T}{8} \qquad S_1(0, t_f) = 4m \frac{X^2}{T} \left(1 - \frac{4\pi^2}{3T^2} \frac{T^2}{64}\right) = 4m \frac{X^2}{T} \left(1 - \frac{\pi^2}{48}\right)$$

[2] I want to consider several possible end times for the trajectory, so n is not fixed yet.

$$= \frac{3.178}{T}\, mX^2 > \frac{\pi}{T} mX^2 = S_0(0,t_f) \tag{14}$$

$$n=1: \quad t_f = \frac{9T}{8} \qquad S_1(0,t_f) = 4m\frac{X^2}{9T}\left(1 - \frac{4\pi^2}{3T^2}\frac{81T^2}{64}\right) = 4m\frac{X^2}{9T}\left(1 - \frac{27\pi^2}{16}\right)$$

$$= -\frac{6.958}{T}\, mX^2 < \frac{\pi}{T} mX^2 = S_0(0,t_f)\,, \tag{15}$$

and as n is increased this action will evidently become smaller and smaller with no lower limit.

These considerations show that, for $n > 0$, $S_0(0,t_f)$ is not the *global* minimum of the action for trajectories between the two points $x(t_i) = 0$ and $x(t_f) = X$. This in itself would not render a principle of least action impossible, because the formulation of the principle requires the trajectories used in the comparison to be small variations of each other, so it would only state that there is a *local* minimum. What the principle states in reality is that the action of the true physical trajectory is a stationary point in comparison with *nearby* (non-physical) trajectories.

Knowledge of a not so close trajectory that has a smaller action than the true trajectory is not useless, however. In fact, we may exploit our result to *construct* nearby trajectories having a lower action than the physical one in the case of sufficiently large time $t_f - t_i$, thus disproving even local minimality.

To this end, call the physical trajectory $x_0(t)$ and let the trajectory described by L_1 be $x_1(t)$. Then consider the family of trajectories given by

$$x(t) = x_\delta(t) = (1-\delta)x_0(t) + \delta x_1(t)\,, \tag{16}$$

where δ is a (small) positive number. Clearly, for $\delta \ll 1$, $x(t)$ gets as close to $x_0(t)$ as we desire. Moreover, $x(t)$ satisfies the initial and final conditions by construction. The Lagrangian corresponding to this trajectory is

$$\begin{aligned}
L &= \frac{m}{2}\dot{x}(t)^2 - \frac{m}{2}\omega^2 x(t)^2 \\
&= \frac{m}{2}\left[(1-\delta)\sqrt{2}X\omega\cos\omega t + \delta\frac{X}{t_f}\right]^2 - \frac{m}{2}\omega^2\left[(1-\delta)\sqrt{2}X\sin\omega t + \delta\frac{Xt}{t_f}\right]^2 \\
&= (1-\delta)^2 L_0 + m(1-\delta)\delta\sqrt{2}X^2\frac{1}{t_f}\left(\omega\cos\omega t - \omega^2 t\sin\omega t\right) + \delta^2 L_1\,.
\end{aligned} \tag{17}$$

In calculating the action, we might anticipate that the integral over the sum of the terms linear in δ must vanish because of the stationarity of the action of the physical trajectory. However, I will not make use of this and just proceed with the calculation. That the term linear in δ goes away, will then serve as an additional check. The action is given by

$$\begin{aligned}
S(0,t_f) &= (1-\delta)^2\int_0^{t_f} L_0\,\mathrm{d}t + \delta^2\int_0^{t_f} L_1\,\mathrm{d}t \\
&\quad + \int_0^{t_f} m\omega(1-\delta)\delta\sqrt{2}X^2\frac{1}{t_f}\left(\cos\omega t - \omega t\sin\omega t\right)\mathrm{d}t \\
&= (1-\delta)^2 S_0(0,t_f) + \delta^2 S_1(0,t_f) \\
&\quad + (1-\delta)\delta\sqrt{2}X^2\frac{m\omega}{t_f}\left(\cancel{\frac{1}{\omega}\sin\omega t} + t\cos\omega t - \cancel{\frac{1}{\omega}\sin\omega t}\right)\Big|_0^{t_f} \\
&= \frac{\pi}{T}mX^2 - 2\delta\frac{\pi}{T}mX^2 + \delta^2\frac{\pi}{T}mX^2 + \delta^2\frac{m}{2}\frac{X^2}{t_f}\left(1 - \frac{1}{3}\omega^2 t_f^2\right) \\
&\quad + \left(\delta - \delta^2\right)\sqrt{2}mX^2\frac{2\pi}{T}\cos\left(\frac{2\pi t_f}{T}\right)\,.
\end{aligned} \tag{18}$$

Now we have $\cos\left(\frac{2\pi t_f}{T}\right) = \cos\left(2\pi\left(n+\frac{1}{8}\right)\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, which means that the terms linear in δ of the first and second lines after the last equal sign of (18) cancel indeed.

We may then rewrite the result as follows:

$$S(0,t_f) - S_0(0,t_f) = -\delta^2 m X^2 \frac{1}{T}\left[\pi - \frac{1}{2n+1/4} + \frac{2\pi^2}{3}\left(n+\frac{1}{8}\right)\right] . \tag{19}$$

Whenever this change of action from the physical trajectory to the modified one is negative, the physical action is not a minimum (it must then be a saddle point or a maximum, because to linear order in δ the action remains unchanged). The sign of its change is decided by the term quadratic in δ. Let us check a few cases:

$$n=0: \qquad S - S_0 = -\delta^2\, mX^2 \frac{1}{T}\left[\pi - 4 + \frac{\pi^2}{12}\right] = 3.594\times 10^{-2}\delta^2\, mX^2\frac{1}{T} > 0 \tag{20}$$

$$n=1: \qquad S - S_0 = -\delta^2\, mX^2 \frac{1}{T}\left[\pi - \frac{4}{9} + \frac{3\pi^2}{4}\right] = -10.10\times \delta^2\, mX^2\frac{1}{T} < 0 \tag{21}$$

$$n=2: \qquad S - S_0 = -\delta^2\, mX^2 \frac{1}{T}\left[\pi - \frac{4}{17} + \frac{17\pi^2}{12}\right] = -16.89\times \delta^2\, mX^2\frac{1}{T} < 0 \tag{22}$$

So we find that the action of the physical trajectory *may* be a local minimum, if $t_f - t_i$ is smaller than a period of the oscillator, but is *not* a local minimum *with certainty*, if that time difference becomes too large, and too large means already larger than a single period.

As it turns out, this is a general pattern. Whenever the action has the form considered here, i.e., $L = T - V$, then it is indeed a local minimum, if the time interval between the endpoints of the trajectory is sufficiently short, and it is only a saddle point, if that time interval becomes too large. The condition for "sufficiently short" is that the final spacetime event occurs before a so-called *kinetic focus event* of the trajectory. There is an interesting article discussing this in some detail on `http://www.scholarpedia.org/article/Principle_of_least_action`. In this article, also a nice argument is given why with a Lagrangian of the form $T - V$ the physical trajectory can never correspond to a local *maximum*. Suppose you leave the trajectory unchanged at its ends but add a piece to it "in the middle" that has a very small amplitude but oscillates strongly, a function of the type $f(t) = \sqrt{\alpha}\sin\frac{t}{\alpha}$, say, with $\alpha \ll 1$, leading to $f(t) \ll 1$ but $\dot{f}(t) = \alpha^{-1/2}\cos\frac{t}{\alpha} \not\ll 1$. This means that the modified trajectory has about the same potential energy as the true physical one (because positions were not changed much) but a much higher kinetic energy (because velocities did change strongly), hence it must have a *larger* action. Since there is always a larger action for a nearby trajectory, the action of the physical trajectory cannot be a maximum. It must be a saddle point, unless it is a minimum. Note that in the case of the harmonic oscillator, the number of trajectories corresponding to saddle points exceeds those corresponding to minima by any measure that assigns larger weight to an infinite time interval than to a finite one.

In relativity, often the action is taken to be given by $mc^2\tau$, a multiple of the proper time. The factor mc^2 is unimportant although it will give the right dimension to the action. Now, in the limit of small velocities and small gravitational fields, this action reduces to the Newtonian one up to a negative factor and a constant shift. Therefore, that relativistic action is a *maximum* for sufficiently short trajectories and may be a saddle point for longer ones, but it can never be a minimum. Hence, the notion of a "principle of least action" is particularly problematic in the relativistic case.

Since I have given only a particular case of the transition from the relativistic action to the Newtonian one in a preceding answer to Stefano Quattrini here on Research Gate, let me indicate the general case now. For weak gravitational fields and small velocities the metric may be written (ϕ is the Newtonian gravitational potential):

$$\begin{aligned} \mathrm{d}s^2 = c^2\mathrm{d}\tau^2 &= c^2\left(1+2\frac{\phi}{c^2}\right)\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \\ &= c^2\left(1+2\frac{\phi}{c^2}\right)\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2\,. \end{aligned} \tag{23}$$

The identification of $\mathrm{d}s^2$ with $c^2\mathrm{d}\tau^2$ is of course possible only for time-like line elements. Note that if we just take the limit of the Schwarzschild metric for small gravitational fields, we will have, in the first line, $\left(1-2\frac{\phi}{c^2}\right)\mathrm{d}r^2$ instead of $\mathrm{d}r^2$. But for small velocities, we can drop the potential term before $\mathrm{d}r^2$, because we have $2\frac{\phi}{c^2}\mathrm{d}r^2 = 2\frac{\phi}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\mathrm{d}t^2 \ll 2\frac{\phi}{c^2}c^2\mathrm{d}t^2$, assuming $\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \leq \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2 \ll c^2$. Setting the action functional equal to $mc^2\tau$, we have

$$\begin{aligned} S(t_i,t_f) &= mc^2\left(\tau_f-\tau_i\right) \\ &= mc^2\int_{t_i}^{t_f}\left\{1+2\frac{\phi}{c^2}-\frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\right\}^{1/2}\mathrm{d}t \\ &\approx mc^2\int_{t_i}^{t_f}\left[1+\frac{\phi}{c^2}-\frac{1}{2c^2}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2\right]\mathrm{d}t \\ &= \int_{t_i}^{t_f}\left[mc^2+m\phi-\frac{m}{2}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2\right]\mathrm{d}t = \int_{t_i}^{t_f}\left[mc^2+V-T\right]\mathrm{d}t \\ &= mc^2(t_f-t_i) - S_{\mathrm{classical}}(t_i,t_f)\,. \end{aligned} \tag{24}$$

The first term[3] does not change under variations of the trajectory, so this definition of the action gives the same equations of motion as the pre-relativistic approach. However, the relativistic action is either a maximum or a saddle point for true trajectories (and never a minimum).

What I have shown here, by way of an example, is that the label “principle of least action” is a misnomer. Moreover, the minimization of the action is not a stability requirement. Otherwise we would have to assume that a single oscillation of a harmonic oscillator is stable but all additional oscillations are unstable, a very strange statement indeed. In fact, we have good approximations of harmonic oscillators at our disposal that display stability of their motion for hundreds and thousands of periodic repetitions. The truth is that the question of whether the action is a true minimum or just a saddle point has nothing to do with stability. This kind of stability considerations is appropriate in thermodynamics where we have statements about the convexity or concavity of energy functionals (the internal energy is a convex function of its natural variables, the entropy a concave one), but not in the case of the action principle.

Other forms of the Lagrangian

Another idea I would like to debunk is the statement that whenever we have well-defined kinetic energy T and potential energy V of the systen, the Lagrangian is given by $L = T - V$. Again a simple example may be used to show that this is false in general.

[3] The additive term mc^2 could be viewed as a modification of the potential energy required by relativity. Then we would still have $L_{\mathrm{class.}} = T - V$, but with the modified potential. And we would have $S = -\int L_{\mathrm{class.}}\mathrm{d}t$.

Consider a charged particle (with charge q) in an electromagnetic field. Let us first note, for the purpose of later use, the relationships between the electromagnetic fields and their potentials:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla\Phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}\,, \qquad \boldsymbol{B}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},t)\,. \tag{25}$$

The kinetic energy of the particle[4] is $T = \frac{m}{2}\dot{\boldsymbol{r}}^2$, its potential energy is $V = q\,\Phi(\boldsymbol{r},t)$, the magnetic field not contributing to it. Let us then set up the Lagrangian

$$\tilde{L} = T - V = \frac{m}{2}\dot{\boldsymbol{r}}^2 - q\,\Phi(\boldsymbol{r},t) \tag{26}$$

and check what we find as equations of motion.

$$\frac{\partial \tilde{L}}{\partial \dot{x}_i} = m\dot{x}_i\,, \qquad \frac{\partial \tilde{L}}{\partial x_i} = -q\frac{\partial \Phi}{\partial x_i}\,, \qquad i = 1\ldots 3\,, \tag{27}$$

from which we get the Euler-Lagrange equations

$$F_i = m\ddot{x}_i = -q\frac{\partial \Phi}{\partial x_i} = q\left(E_i(\boldsymbol{r},t) + \frac{\partial A_i(\boldsymbol{r},t)}{\partial t}\right)\,. \tag{28}$$

Is this the correct form of the Lorentz force? Only in the absence of a magnetic field. In that case, we may take $\boldsymbol{A} = 0$ and Eq. (28) reduces to $\boldsymbol{F} = q\boldsymbol{E}$ which is the correct electrostatic limit.

However, for a general electromagnetic field (28) is not the correct force expression. We know that the Lorentz force is given by

$$F_i = qE_i + q\left(\dot{\boldsymbol{r}}\times\boldsymbol{B}\right)_i\,, \tag{29}$$

$$F_i = q\left[-\frac{\partial \Phi}{\partial x_i} - \frac{\partial A_i}{\partial t} + \dot{x}_k\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right)\right]\,, \tag{30}$$

where we have used the Einstein summation convention in the component form and employed $(\dot{\boldsymbol{r}}\times\boldsymbol{B})_i = \epsilon_{ikl}\dot{x}_k\epsilon_{lmn}\frac{\partial A_n}{\partial x_m} = (\delta_{im}\delta_{kn} - \delta_{in}\delta_{km})\dot{x}_k\frac{\partial A_n}{\partial x_m} = \dot{x}_k\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right)$. The force given by Eq. (30) is clearly different from that described by Eq. (28).

Nevertheless, there is a Lagrangian for a particle in an electromagnetic field and its equations of motion follow from Hamilton's principle, i.e., the standard action principle. That Lagrangian is given by

$$L = \frac{m}{2}\dot{\boldsymbol{r}}^2 - q\,\Phi(\boldsymbol{r},t) + q\,\dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r},t)\,. \tag{31}$$

From this, we obtain the canonical momentum

$$\begin{aligned} p_i \equiv \frac{\partial L}{\partial \dot{x}_i} &= m\dot{x}_i + qA_i\,, \\ \boldsymbol{p} &= m\dot{\boldsymbol{r}} + q\boldsymbol{A}(\boldsymbol{r},t)\,, \end{aligned} \tag{32}$$

i.e., the canonical momentum is different from the kinetic one, which is given by $\boldsymbol{p}_{\mathrm{kin}} = m\dot{\boldsymbol{r}}$. In fact, it is defined only up to a gauge degree of freedom, since we can choose the divergence of $\boldsymbol{A}$ freely, so the second summand is not fixed before fixing the gauge. Note that it is this canonical momentum which enters the Heisenberg uncertainty relations after quantization. Contrary to it, the kinetic momentum is well-defined and directly measurable.

[4]We consider the non-relativistic limit of small velocities.

Let us check on the equations of motion following from the action principle with L given by Eq. (31). We have $\frac{\partial L}{\partial x_i} = -q\frac{\partial \Phi}{\partial x_i} + q\,\dot{\boldsymbol{r}}\cdot\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial x_i}$, whence

$$
\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} &= m\ddot{x}_i + q\frac{\mathrm{d}A_i}{\mathrm{d}t} + q\frac{\partial \Phi}{\partial x_i} - q\,\dot{x}_k\frac{\partial A_k}{\partial x_i} \\
&= m\ddot{x}_i + q\frac{\partial A_i}{\partial t} + q\,\dot{x}_k\frac{\partial A_i}{\partial x_k} + q\,\frac{\partial \Phi}{\partial x_i} - q\,\dot{x}_k\frac{\partial A_k}{\partial x_i} \\
&= m\ddot{x}_i - q\Bigg[\underbrace{-\frac{\partial \Phi}{\partial x_i} - \frac{\partial A_i}{\partial t}}_{E_i} + \underbrace{\dot{x}_k\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right)}_{(\dot{\boldsymbol{r}}\times\boldsymbol{B})_i}\Bigg] = m\ddot{x}_i - F_i\,.
\end{aligned}
\tag{33}
$$

Obviously, this Lagrangian leads to the correct equations of motion and reproduces the Lorentz force. It is however not given by $T - V$, rather we have $L = T - V^*$, where

$$
V^* = q\,\Phi(\boldsymbol{r},t) - q\,\dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r},t) \tag{34}
$$

is a so-called *generalized potential.* A generalized potential may be velocity dependent and its defining property is that the force can be calculated from it according to

$$
F_i = -\frac{\partial V^*}{\partial x_i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V^*}{\partial \dot{x}_i}\,. \tag{35}
$$

Note that the *Hamiltonian* does not contain V^* but rather V. The Hamiltonian is defined, as usual, via a Legendre transformation of the Lagrangian:

$$
H = \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L = (m\dot{\boldsymbol{r}} + \cancel{q\boldsymbol{A}})\cdot\dot{\boldsymbol{r}} - \frac{m}{2}\dot{\boldsymbol{r}}^2 + q\,\Phi - \cancel{q\,\dot{\boldsymbol{r}}\cdot\boldsymbol{A}} = \frac{m}{2}\dot{\boldsymbol{r}}^2 + q\,\Phi\,, \tag{36}
$$

and this is obviously the total energy, so $H = T + V$.[5] Hence, we can say that we have a system with energy conservation, in which kinetic and potential energy can be formulated and the Hamiltonian is given by their sum. Nevertheless, the Lagrangian is not given by their difference.

Finally, it should be mentioned that $L = T - V^*$ is not the most general expression for a Lagrangian. Neither is the standard form of the relativistic Lagrangian for a particle in a gravitational field of this form,[6] nor is the Einstein-Hilbert action[7] an integral over a Lagrangian (density) of this particular form. In field theory, the form of Lagrangian densities is often postulated from symmetry arguments and the equations of motion are then obtained without a direct identification of kinetic energy and potential energies. A Hamiltonian density can then be derived and it typically allows to identify analogs of the kinetic and potential energies.

[5] To use the Hamiltonian equations of motion, we would rewrite it in terms of $\boldsymbol{p}$ and $\boldsymbol{r}$, i.e. $H = \frac{1}{2m}\left(\boldsymbol{p} - q\boldsymbol{A}\right)^2 + q\,\Phi$.

[6] I don't know how to rewrite the relativistic action, i.e., the proper time (times mc^2), as an integral of the difference between kinetic energy and generalized potential, except in the limit of weak fields and small velocities.

[7] From which the field equations of general relativity are derived.