Revisiting real functions having a constant limit as $x \to \infty$. What can we say about the limiting behavior of their derivative?

Klaus Kassner 12 September 2021

I already wrote an essay on one aspect of this question near the end of 2016. Its trigger was a ResearchGate question asked by Muhammet Ali Okur:
https://www.researchgate.net/post/If_a_functions_limit_is_zero_at_infinity_does_that_imply_its_derivative_has_same_limit_at_infinity#view=5cf2da2bf0fb6213f01c5013

We may extend the original question to: If the limit of a real function at infinity is a constant, does that imply that its derivative has the limit zero at infinity? This was answered in the negative pretty fast, and my essay referred to the problem whether adding the condition that the function is of bounded variation will lead to a positive answer. I gave a sequence of counterexamples $f_k(x)$ showing that even with this additional condition, $\lim_{x\to\infty} f_k'(x)$ does not exist. My examples were not only differentiable but analytic on the whole real line (in fact, they were even analytic on $\mathbb{C}$), so their derivative existed everywhere on $\mathbb{R}$. Later (in May 2019), J. Domsta gave a counterexample that was even somewhat simpler to treat. Because it was defined as an integral, the derivative could be read off without any effort (whereas in my case its calculation is not difficult, but still requires a minimum of care).

I also gave a simple condition that, when added to the assumptions about the function, indeed leads to the limit of its derivative for $x \to \infty$ to vanish. This condition is simply existence of the limit. If $\lim_{x\to\infty} f'(x)$ exists and $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$, then we also have $\lim_{x\to\infty} f'(x) = 0$. This I stated without proof.

Recently, Itzhak Barkana posted a paper on ResearchGate which can be found at
https://www.researchgate.net/publication/333561851_If_a_function_ft_has_a_constant_limit_for_t_does_its_derivative_always_have_to_end_with_a_constant_limit_of_0_for_t
and in which he claims that the counterexamples are wrong and that indeed the limit of $f'(x)$ as $x \to \infty$ must be zero if $f(x)$ tends to a constant value in the same limit. So he essentially denies the correctness of the proofs given before.

The purpose of the present essay is, on the one hand, to explicitly give the proof for the aforementioned proposition that I only stated in my earlier discussion of the subject. On the other hand, I will analyse Barkana's approach and point out what is wrong with it. For those who are impatient, besides a number of peripheral errors (that I may mention or skip) his main error is an insufficient distinction between the limit of a function as its argument approaches x_0 and its value at x_0 (these can be different!), together with a negligence of the fact that ∞ is not a number in $\mathbb{R}$. However, the main reason why I am discussing this at all (instead of considering his paper completely useless), is that his approach may actually be useful in non-standard analysis on the extended real line $\mathbb{R}\cup\{\infty\}$. Needless to say that if I were a referee of the paper, I would still require major revision...

We start with the positive assertion **P1**:
Let $f(x)$ be a real function on the real numbers, satisfying $\lim_{x\to\infty} f(x) = c$, with $c \in \mathbb{R}$ a constant, and its derivative $f'(x)$ having the property that $\lim_{x\to\infty} f'(x)$ exists. Then it is true that $\lim_{x\to\infty} f'(x) = 0$.[1]

[1]In my proofs, I will not normally give premises in terms of very detailed mathematical notation, such as

Proof:
If $\lim_{x\to\infty} f'(x)$ exists, it must take some value $d \in \mathbb{R}$. Assume now that $d \neq 0$. As we shall see, this leads to a contradiction.

Set $g(t) = f(x)$, $t = 1/x$. The preimage of a small neighborhood of $t = 0$ with $|t| < \delta$, $\delta > 0$ contains all values x with $|x| > 1/\delta$ and may be considered a "neighborhood of infinity". The mapping is continuous for $x \neq 0$. We have

$$g'(t) = f'(x)\frac{\mathrm{d}x}{\mathrm{d}t} = f'(x)\left(-\frac{1}{t^2}\right) , \tag{1}$$

$$f'(x) = -g'(t)t^2 ,$$

$$d = \lim_{x\to\infty} f'(x) = \lim_{t\to 0} -g'(t)t^2 \qquad \Longleftrightarrow \qquad g'(t) \sim \frac{-d}{t^2} , \quad t \to 0 . \tag{2}$$

Here, I introduce the notation $\sim$ from asymptotic analysis. Its precise meaning is given by $a(t) \sim b(t)$, $t \to t_0 \;\Leftrightarrow\; \lim_{t\to t_0} a(t)/b(t) = 1$. This is the meaning of the symbol, whenever the right-hand side is a *function*. It then represents a single asymptotic relationship. If the right-hand side is a series, the symbol has a somewhat different meaning, representing infinitely many asymptotic relationships. We will not encounter that case here. Another symbol that I will use is $\ll$, with the meaning $a(t) \ll b(t)$, $t \to t_0 \;\Leftrightarrow\; \lim_{t\to t_0} a(t)/b(t) = 0$.[2]

From (2), we conclude that a function $h(t)$ exists[3] that satisfies

$$g'(t) = \frac{-d}{t^2} + h(t) , \qquad h(t) \ll \frac{d}{t^2} , \quad t \to 0 \tag{3}$$

and integrating we obtain

$$g(t) = \frac{d}{t} + \int^t h(t')\mathrm{d}t' + \text{const.} , \qquad \int^t h(t')\mathrm{d}t' \ll \frac{d}{t} , \quad t \to 0 . \tag{4}$$

The smallness relation for the integral of h holds, because the integrand $-d/t^2$ of the first integral does not change sign and $h(t)$ is small in comparison with that integrand, so even the integral of $|h(t)|$ remains small in comparison with d/t as $t \to 0$, because that term diverges and so any constant contribution from the lower boundary of the integral cannot become large enough to overcome the dominant term d/t in the limit $t \to 0$. Hence we have

$$g(t) \sim \frac{d}{t} , \qquad t \to 0 , \tag{5}$$

implying

$$f(x) \sim d\,x , \qquad x \to \infty , \tag{6}$$

which because of $d \neq 0$ contradicts the fact that $f(x) \sim c$, $x \to \infty$ by assumption. Hence, we cannot have $d \neq 0$, which proves that $\lim_{x\to\infty} f'(x) = 0$. ■

$f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$. That makes them less readable. While a mathematician may consider the omission a lack of rigor (because some premises may not be stated explicitly enough), I think that the readers will be able to fill in "self-understood" points. Also, it will be seen easily that and how the proofs can be made rigorous with little effort, for anyone who is interested in getting hold of a watertight formulation.

[2] Note that the relationsship $a(t) \ll b(t), t \to t_0$ can hold for $a(t) > 0$, $b(t) < 0$. The symbols are read $a(t)$ *is asymptotic to* $b(t)$ *for* $t \to t_0$ and $a(t)$ *is very small in comparison with* $b(t)$ *as* $t \to t_0$, respectively.

[3] That function is simply $h(t) = g'(t) + d/t^2$.

The astute reader may note a gap in this proof. I introduce a new function $h(t)$ that I integrate without having checked whether it is integrable. This is because I did not want to overload the proof with distracting details. That $h(t)$ is integrable, can be concluded from the existence of $\lim_{x\to\infty} f'(x)$. The idea is as follows. The existence of the limit requires, by the very definition of a limit (which I will give below) $f'(x)$ to exist for all x greater than some value x_0 (to be found in each concrete case). Since $f'(x)$ is a derivative, it is integrable from that value x_0 (or any larger value) to arbitrarily large x. Via the transformation law given by Eq. (1), $f'(x)\,\mathrm{d}x = g'(t)\,\mathrm{d}t$, we infer that $g'(t)$ is integrable on corresponding intervals from $t_0\,(= 1/x_0)$ towards arbitrarily *small* $t > 0$, and by construction its antiderivative is $g(t)$. Hence, the first two terms of Eq. (3) are integrable from t_0 to t, so their difference $h(t)$ must be integrable, too. If the lower bound of integration is taken to be t_0, the const. in Eq. (4) becomes $g(t_0) - d/t_0$, which is negligibly small in comparison with d/t for $t \ll t_0$. The same holds for the integral on $h(t)$ and so we arrive at Eq. (5).

Next, I will give, for reference, a counterexample to the statement that $\lim_{x\to\infty} f'(x) = 0$, if all the conditions of **P1** except for the requirement of existence of the limit are satisfied.[4] I will discuss one of the basic objections by Barkana and show that it does not invalidate the counterexample, so the proposition that he claims to prove in his paper is wrong. Finally, I will consider the strategy of his proof and demonstrate what he actually proved.

Consider the function

$$f(x) = \frac{\sin x^2}{x}\,, \tag{7}$$

defined for all $x \in \mathbb{R}$, $x \neq 0$. It is seen by inspection, that $\lim_{x\to\infty} f(x) = 0$ – the absolute value of the numerator is bounded by 1, so we will have $|f(x)| \le 1/x$, $\forall x \in \mathbb{R}$, which means that $f(x)$ must go to zero as x grows without bound.

The derivative of $f(x)$ is

$$f'(x) = 2\cos x^2 - \frac{\sin x^2}{x^2}\,. \tag{8}$$

Now Barkana objects to using this formula for $x \to \infty$, arguing that $\sin x^2$ is not differentiable for $x \to \infty$, becoming a bunch of spikes there.[5] Now this is a somewhat funny objection, as there is – so far – no notion of being "differentiable for $x \to \infty$". We only have a definition for differentiability at a point. And $\sin x^2$ is of course differentiable at *all* values of x.[6] That is, if we wish "differentiable for $x \to \infty$" to mean "differentiable for all $x > x_0$ with x_0 taken sufficiently large", then $\sin x^2$ is also "differentiable for $x \to \infty$". But that is obviously not what Barkana wants his notion of differentiability to mean, as he explicitly states $\sin x^2$ not to be "differentiable for $x \to \infty$".[7]

The function $f(x)$ itself is differentiable for all $x \neq 0$. Moreover, it is certainly differentiable for "arbitrarily large but finite" x, to use an expression repeatedly used by Barkana in trying to

[4] This simple example is the one that lead to the answer with most recommendations in the ResearchGate thread.
[5] This is because the oscillations of the function happen with ever shortening local wavelength.
[6] It is an entire function, after all. For the experts.
[7] If he means that the limit of the derivative of $\sin x^2$ does not exist for $x \to \infty$, then he is right, but it is insufficient to refer to it becoming a bunch of spikes there. The function $\sin x^2/x^2$ also becomes a bunch of spikes as $x \to \infty$, but its derivative has a limit there.

explain what mathematicians mean by the notion of limit. His explanation is not quite correct, therefore I will give here the nominally rigorous definitions for the meaning of the limit of a function as $x \to x_0$ and as $x \to \infty$.[8] (In order to avoid clumsiness, I do not specify explicitly in the formulation that all numbers considered are real numbers, therefore the restriction *nominally* regarding the rigor of the definitions. They are rigorous with a few preliminaries.)

Definition **D1**: The limit of $g(x)$ as x approaches x_0 is r, written $\lim_{x \to x_0} g(x) = r$, if for any $\varepsilon > 0$, there exists a $\delta > 0$ so that $|g(x) - r| < \varepsilon$ for all x satisfying $0 < |x - x_0| < \delta$.

Definition **D2**: The limit of $g(x)$ as x approaches ∞ is r, written $\lim_{x \to \infty} g(x) = r$, if for any $\varepsilon > 0$, there exists a $\delta > 0$ so that $|g(x) - r| < \varepsilon$ for all x satisfying $|1/x| < \delta$.[9]

A few remarks may be in order. First, these definitions do not require evaluating g at the argument, for which the limit is desired. In the first definition, this follows from the restriction $0 < |x - x_0|$, in the second from the restriction that x has to be a number (which ∞ is not).[10] Second, talking about a limit "as x approaches" some value or "for x approaching" some value, suggests an idea of motion that is not implied by the definition. Clearly, this notion is useful in the visualization of limits as "processes" and visualization may be important to clarify ideas before delving into a calculation. Nevertheless, a statement about a limit is simply a statement about a property shared by the function values of a set of points. (All points, for which x is closer to x_0 than δ have the property that $f(x)$ is closer to r than the distance given by ε, except possibly the point $x = x_0$ itself.) Third, the δ appearing in the definition usually depends on ϵ. So if we wish to determine, starting from the definition, whether r is a limit of g, what we typically do is to prescribe ε and to try and find a δ so that the closeness property is satisfied. If we manage to to this for arbitrary $\varepsilon > 0$, then we confirm the limit property, if we can prove that no such δ exists, we have shown that r is not a limit. (To prove that the limit does not exist, we have to demonstrate that *no* r value will do.)

In light of these precise definitions, the notion of being "differentiable for $x \to \infty$" remains murky, suggesting differentiability in a set of more than one point at the same time, i.e., with the same limit for all of them.[11] Probably this is not what Barkana means. In any case, it is not what follows from his formulas. There the interpretations is clear, and one purpose of this article is to give that clarification.

Let us return to the objection of evaluating $f'(x)$ according to Eq. (8), i.e., using the product and chain rules. Barkana suggests to use the definition of a derivative instead:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \,. \tag{9}$$

Now there are certainly instances where recourse must be had to this formula, defining what is meant by a derivative. But the rules that were used in evaluating $f'(x)$ in Eq. (8) from $f(x)$ in Eq. (7) are *derived* from formula (9), given certain regularity conditions for the constituent

[8] Two different definitions are needed, because the notion of a neighborhood is different for finite x values and for "the point ∞".

[9] There is a similar definition for x approaching $-\infty$.

[10] Therefore, this kind of limit is specified as *deleted* limit. For an *undeleted* limit, the restriction $0 < |x - x_0| < \delta$ in **D1** is replaced by $|x - x_0| < \delta$, which has the consequence that if $g(x_0) \neq r$, g cannot have the limit r at x_0. With a deleted limit (the normal case), it is possible for the limit of g, as x approaches x_0, to be different from $g(x_0)$. This happens for functions g that are discontinuous at x_0.

[11] We can have a limit of a derivative for $x \to \infty$ but differentiability is a property that is defined for each point separately (without implying differentiability for the limit automatically).

functions (essentially continuity of the functions and of their derivatives).[12] Hence, there can be no doubt that Eq. (8) gives the correct derivative $f'(x)$ corresponding to $f(x)$ of Eq. (7) on the full real axis diminished by the point $x = 0$. And indeed, Barkana finds the same result by explicit use of Eq. (9).

But then we can check whether the derivative has a limit for $x \to \infty$ by direct exploitation of definition **D2**. This is certainly not the fastest way, but it has the advantage of not being affected by any argument of *how to calculate* the limit of the derivative for $x \to \infty$. To check the definition, we never need anything but finite arguments of f' and for finite arguments, we have a completely valid form. To proceed, we need a value r for the presumed limit. Since $f'(x)$ takes on the value $+2$ for $x^2 = 2n\pi$ where $n \neq 0$ is an integer, we will first assume $r = 2$ and, having shown that this is not the limit, we will consider the case $r \neq 2$.

Proof of nonexistence of $\lim_{x\to\infty} f'(x)$:
Set $r = 2$ and $\varepsilon = 1$. If r is the sought-for limit, then there has to exist a $\delta > 0$, so that we have

$$\left|f'(x) - r\right| = \left|2\cos x^2 - \frac{\sin x^2}{x^2} - 2\right| < \varepsilon = 1\,, \text{for all } x > 1/\delta\,. \tag{10}$$

However, setting $x = ((2n+1)\pi)^{1/2}$ with $n \in \mathbb{N}$, we find $\cos x^2 = -1$ and $\sin x^2 = 0$, hence

$$\left|2\cos x^2 - \frac{\sin x^2}{x^2} - 2\right| = |-2 - 0 - 2| = 4 > 1\,, \tag{11}$$

and since we can always choose n so large that $x > 1/\delta$ no matter what value is taken for δ, there is no possible choice of δ that makes $|f'(x) - r|$ smaller than 1 for *all* values of $x > 1/\delta$. This demonstrates that $r = 2$ cannot be the limit of $f'(x)$, as x approaches infinity.

Now take r arbitrary but $r \neq 2$. Set $\varepsilon = |1 - r/2|$. This is greater than 0. Setting $x = (2n\pi)^{1/2}$, we have $\cos x^2 = 1$ and $\sin x^2 = 0$, hence

$$\left|f'(x) - r\right| = \left|2\cos x^2 - \frac{\sin x^2}{x^2} - r\right| = |2 - r| > \frac{1}{2}\,|2 - r| = |1 - r/2| = \varepsilon. \tag{12}$$

Again, no matter what value we try for δ, there are always x values larger than $1/\delta$ (obtainable by choosing n large enough) for which $|f'(x) - r|$ does not become smaller than ε. This demonstrates that $r \neq 2$ cannot be the limit of $f'(x)$ either, as x approaches infinity.

Hence, there is no possible limit, which demonstrates its nonexistence. ■

Note that Barkana's objection does not affect this proof at all. Even if he were right that he has a more correct way (than the other participants in the ResearchGate discussion) of calculating the limit of the derivative as $x \to \infty$ (and this gives zero), this does not matter. We have tried out all possible limits[13] (including zero) and found that none of them satisfies the definition that would make it a limit.

[12] As an example, I sketch the derivation of the product rule: $(u(x)v(x))' = \lim_{h\to 0} \frac{u(x+h)v(x+h)-u(x)v(x)}{h} = \lim_{h\to 0} \frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h} = \lim_{h\to 0} \left[\frac{(u(x+h)-u(x))}{h} v(x+h) + u(x)\frac{(v(x+h)-v(x))}{h}\right] = u'(x)v(x) + u(x)v'(x)$, where to obtain the final equality the fact has been used that a limit of a sum/product is the sum/product of the corresponding limits of the summands/factors, provided that each of these limits exists separately. The latter condition is secured if u, v, u' and v' are all continuous at x.

[13] $\{r | r \in \mathbb{R}, r = 2 \text{ or } r \neq 2\}$ exhausts all possible real values for r.

Now it is time to consider Barkana's suggestion of how the limit of the derivative of $f(x)$ should be calculated for $x \to \infty$. Since he discusses cases for finite x first, I will write the general expression and call it $\mathcal{E}_1$ for further reference:

$$\mathcal{E}_1[x_0] = \lim_{h\to 0} \lim_{x\to x_0} \frac{f(x+h)-f(x)}{h} . \tag{13}$$

Here, $x_0 = \infty$ in the case that we are finally interested in, but it is very useful to start with considering finite values of x_0.

For a quick interpretation, we note that *provided f is continuous at x_0 and at x_0+h*, the inner limit simply gives $(f(x_0+h)-f(x_0))/h$, i.e., the difference quotient (9) at x_0 and hence the outer limit will give $f'(x_0)$, if it exists. Note that no *limit* of $f'(x)$ is calculated, only the *single value* $f'(x_0)$.

For $x_0 = \infty$, we cannot argue this way, because neither the function $f(x)$ nor $f'(x)$ are defined at infinity. However, as we shall see later, this can be cured in non-standard analysis, by extending the domain of the function to the projectively extended real line[14] $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ or the affinely extended real line[15] $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$. Later, we shall discuss in more detail how non-standard analysis makes sense of $\mathcal{E}_1[\infty]$.

I did not follow the ReseachGate discussion of this question closely after 2016. However, I was contacted via E-mail by Itzhak Barkana, who asserted that the had "solved" the question in a way that contradicted the standard answer and written a paper on it. I pointed out to him that in order for an expression in the style of $\mathcal{E}_1$ to represent a limit of the derivative of f, the limits in general have to be taken in the opposite order. And I think I saw later that some of the mathematically well-versed members of the ResearchGate community told him the same thing. That is, we are certain that the expression to be considered is, instead of $\mathcal{E}_1$ the following:

$$\mathcal{E}_2[x_0] = \lim_{x\to x_0} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} . \tag{14}$$

Here, the inner limit gives, if it exists, the derivative $f'(x)$ and the outer limit produces, *if $f'(x)$ is continuous at $x = x_0$*, the value $f'(x_0)$.

Hence, if the two limits commute and both f and f' satisfy certain continuity conditions, they give the same result. But only $\mathcal{E}_2$ even *considers* a limit of the derivative function f'. Expression $\mathcal{E}_1$ considers a limit of difference quotients and then takes the limit defining (in some cases) a derivative of f in a single point.

Barkana claimed that he never interchanged orders of limits, which I first interpreted as him saying that he did not even take the limit $x \to \infty$, But he clearly does, as he gets the result that the difference quotient becomes zero. While $f(x+h) - f(x)$ may happen to be zero (at isolated points) for large but finite x, this result is not independent of h and taking the limit $h \to 0$ afterwards will make the difference nonzero again (in fact, it will oscillate for our example from Eq. (7)). In order to get zero *independent* of h, the limit $x \to \infty$ must actually

[14] This is an open set and the point ∞ is approachable along the positive half line as well as along the negative half line. We could consider this a subset of the extended complex plane, which is obtained from the complex plane via addition of the single point infinity. Any path leading to infinity leads to that point, so $-\infty$ and ∞ are identical.

[15] This is a closed set, and functions can only have right-sided derivatives at $-\infty$ and left-sided derivatives at ∞.

be taken. Meanwhile, I think that Barkana did not want to say that he did not take two limits but that he did not interchange their sequence, because he thinks that his sequence of limits is the correct one.[16]

Before embarking on a precise discussion of what the two expressions $\mathcal{E}_1$ and $\mathcal{E}_2$ actually mean, I would like to give an informal argument that is not rigorous but uses notions that may be more convenient to readers who have a good deal of working experience with real functions and are able to develop a decent intuitive feeling about their behavior but may not be too comfortable with abstract mathematical proofs based on logic without visualization.[17]

First, we note that $\sin x^2$ is an oscillatory function and we can define a local wavelength of the oscillation as follows. Suppose $\sin x_1^2 = a$, then we take as the wavelength the distance to the next value x_2 ($> x_1$), for which $\sin x_2^2 = a$ *and* the passage through the value a is in the same sense, i.e., if $\sin x^2$ passes through a from below (above) at x_1, it should do the same at x_2. Obviously, we do not need this directional information, if a is either a maximum or a minimum, because those will be taken on only once during each oscillation (whereas all other function values will be taken on twice during each oscillation). Then we have:

$$x_2^2 - x_1^2 = 2\pi \qquad \Rightarrow \qquad \lambda(x_1) \equiv x_2 - x_1 = \frac{2\pi}{x_1 + x_2} < \frac{\pi}{x_1}\,, \tag{15}$$

i.e., the local wavelength becomes smaller as x_1 increases (and goes to zero for $x_1 \to \infty$). Next, the *difference quotient* $(f(x+h) - f(x))/h$ has the geometrical meaning that it is the slope of a secant to the graph of f connecting the points $(x, f(x))$ and $(x+h, f(x+h))$. And finally, the derivative of $f(x)$ at x (or its differential quotient) is the slope of the tangent to the graph of f at the point $(x, f(x))$. The definition (9) then simply comes from the observation that the slope of the secant becomes a better and better approximation to the slope of the tangent as h gets smaller. So we calculate derivatives by improving approximations via the slope of secants.

Barkana introduces the inner limit of $\mathcal{E}_1[\infty]$ by saying that he sets h arbitrarily small but fixed and then lets x become arbitrarily large. Now, our difference quotient may have been a good approximation to the derivative at the initial value of x. But as x gets larger, two things happen. First, $f(x+h) - f(x) = \frac{\sin(x+h)^2}{x+h} - \frac{\sin x^2}{x}$ get smaller, because the numerators are bounded by one and the denominators get larger. Second, the local wavelength $\lambda(x)$ also gets smaller and will eventually become smaller than h. So the difference quotient becomes arbitrarily small, but the interval, over which the secant is taken contains more and more wavelengths. But we know from Eq. (8) that during each full oscillation, the derivative $f'(x)$ runs through all the values between -2 and $+2$.[18] So the difference quotient is not a good approximation to the differential quotient (i.e., the derivative) anymore when h is larger than the local wavelength. Therefore, there is no reason to expect that the outer limit, taken when the inner one has become independent of h will approximate *any* derivative!

[16]Maybe he also referred to the fact that once the inner limit was taken, the outer was not needed anymore, because the result was independent of h and so zero was obtained even without taking the outer limit. While that is true, the outer limit must formally be taken, in order to get a derivative. Otherwise, all one has shown is that the difference $f(x+h) - f(x)$ goes to zero for $x \to \infty$, which is trivial given that f has a finite limit, as x approaches infinity.

[17]In my experience, a number of engineers and experimental physicists belong into that category.

[18]At least. The extrema of $f'(x)$ are not exactly at $x^2 = (2n+1)\pi$, $n \in \mathbb{N}$, but the values -2 and $+2$ are taken on by $f'(x)$ during each full oscillation of $\sin x^2$.

Note that the argument does not work for finite x_0. Since the fixed value of h for the inner limit of (13) can be taken arbitrarily small, we may choose it to be much smaller than $\lambda(x_0)$ which is the smallest wavelength that will appear for $x \leq x_0$. But then the difference quotient is a good approximation to the derivative throughout the interval of x considered (with upper limit x_0), and taking the limit $h \to 0$ will then give the derivative. For infinite x_0, however, no matter how small we choose h, $\lambda(x)$ will at some point become smaller than the finite value of h, so the difference quotient loses its meaning as an approximation to the derivative.

To get a better idea on what happens in performing the limits $\mathcal{E}_1[x_0]$ and $\mathcal{E}_2[x_0]$ and to go beyond the above informal argument, let us consider a few examples with finite x_0. We start with the simple case of an analytical function.

Set $f(x) = \sin x$, then we have $f'(x) = \cos x$, and

$$\mathcal{E}_1[0] = \lim_{h\to 0}\lim_{x\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1\,, \tag{16}$$

$$\mathcal{E}_2[0] = \lim_{x\to 0}\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{x\to 0} \cos x = 1\,. \tag{17}$$

No surprises here, the limits commute, we get the same result for both expressions.

Now consider

$$u(x) = \begin{cases} x(1 + x\sin(1/x)) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{18}$$

(and replace f by u in the expressions $\mathcal{E}_1$ and $\mathcal{E}_2$), then we have, for $x \neq 0$,

$$u'(x) = 1 + 2x\sin(1/x) - \cos(1/x)\,. \tag{19}$$

We immediately see that the limit of $u'(x)$ does not exist for $x \to 0$. But the derivative at $x = 0$ exists! We have $\lim_{h\to 0}(u(h) - u(0))/h = \lim_{h\to 0}(1 + h\sin(1/h)) = 1$. Hence, the derivative can be written as

$$u'(x) = \begin{cases} 1 + 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}\,. \tag{20}$$

Clearly, $u'(x)$ is discontinuous at $x = 0$, because $u'(0)$ exists, but $\lim_{x\to 0} u'(x) \neq u'(0)$. Let us denote this inequality coming from nonexistence of the limit by the following suggestive notation:

$$u'(0) = 1 \neq \lim_{x\to 0} u'(x) = \langle\,\nexists\,\rangle\,. \tag{21}$$

We are now in a position to evaluate our expressions $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\mathcal{E}_1[0] = \lim_{h\to 0}\lim_{x\to 0} \frac{u(x+h) - u(x)}{h} = \lim_{h\to 0} \frac{u(h) - u(0)}{h} = \lim_{h\to 0} \frac{h(1 + h\sin(1/h))}{h} = 1\,, \tag{22}$$

$$\mathcal{E}_2[0] = \lim_{x\to 0}\lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = \lim_{x\to 0}\left(1 + 2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) = \langle\,\nexists\,\rangle\,. \tag{23}$$

So we find that $\mathcal{E}_1$ gives us the *value* of the derivative of $u(x)$ at $x = 0$, whereas $\mathcal{E}_2$ indicates non-existence of the *limit* of the derivative $u'(x)$, as x approaches 0. From both expressions together, we can conclude that the derivative is not continuous at $x = 0$. (Note that the ResearchGate question never was about the *value* of the derivative $f'(\infty)$ but about the *limit* of $f'(x)$ as $x \to \infty$.)

Here is another interesting case. Sometimes we obtain solutions to equations as limit expressions themselves. Consider therefore the one-parameter family of functions

$$v_\mu(x) = \frac{\sin x}{x + \mu^2}\,, \quad \mu \in \mathbb{R} \tag{24}$$

and assume that the solution of our (physical) problem is given by the limit

$$v(x) = \lim_{\mu \to 0} v_\mu(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}\,. \tag{25}$$

Note that $\lim_{x\to 0} v(x) = 1$, so $v(x)$ has a discontinuity at $x = 0$. This means that it cannot be differentiable at $x = 0$. However $\lim_{x\to 0} v'(x)$ does exist, as we shall see.[19] Let us then calculate the derivative function $v'(x)$;

$$\begin{aligned} v'(x) &= \frac{x\cos x - \sin x}{x^2}\,, \qquad x \neq 0 \\ v'(0) &= \lim_{h\to 0} \frac{v(h) - 0}{h} = \lim_{h\to 0} \frac{\sin h}{h^2} = \lim_{h\to 0} \frac{\cos h}{2h} = \langle \nexists \rangle\,, \\ \lim_{x\to 0} v'(x) &= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x} = \lim_{x\to 0} -\frac{\sin x}{2} = 0\,, \end{aligned} \tag{26}$$

where we have used l'Hôpital's rule to evaluate the limits in lines two and three. If the limit in the second line were one-sided, it would give ∞ or $-\infty$, but since h can have any sign in a proper limit, we cannot state more than that the derivative is undefined.

On to our two double limit expressions:

$$\mathcal{E}_1[0] = \lim_{h\to 0}\lim_{x\to 0} \frac{v(x+h) - v(x)}{h} = \lim_{h\to 0} \frac{v(h)}{h} = \lim_{h\to 0} \frac{\sin h}{h^2} = \langle \nexists \rangle\,, \tag{27}$$

$$\mathcal{E}_2[0] = \lim_{x\to 0}\lim_{h\to 0} \frac{v(x+h) - v(x)}{h} = \lim_{x\to 0} v'(x) = 0\,. \tag{28}$$

Again, expression $\mathcal{E}_1$ gives us the *value* of the derivative at $x = 0$, which here means it becomes undefined, whereas expression $\mathcal{E}_2$ gives us the *limit* of the derivative, as x approaches zero, which is perfectly well-defined here, because $v(x)$ has a removable singularity at $x = 0$. If we replace $v(x)$ by $\tilde{v}(x) = \begin{cases} v(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$, we obtain a function that is differentiable (even infinitely often) at $x = 0$. Since the *value* of the derivative at $x = 0$ plays no role at all in the definition of its *limit* as x approaches 0,[20] its limit must be the same for $v(x)$ and $\tilde{v}(x)$.

Finally, let us consider a wild example. There exist functions that are *everywhere continuous* on the real axis but *nowhere differentiable*.[21] An example is the Weierstrass function:

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right)\,. \tag{29}$$

[19] The situation is opposite to that in the preceding example. There, the derivative exists at $x = 0$, but its limit for $x \to 0$ does not, whereas here, the derivative does not exist, but its limit for $x \to 0$ does.

[20] Remember, we are dealing with the standard definition of limit, which is the deleted limit.

[21] It should be clear that such a function cannot be drawn. So we cannot argue with the graph of the function but must rely on logic entirely.

This is actually a two-parameter family of functions. It was shown by Weierstrass to have the above-mentioned properties for certain parameter values a and b (*viz.* $0 < a < 1$ and b being restricted to odd integers greater than 5) and later by Hardy [1] to be everywhere continuous and nowhere differentiable, provided only that $0 < a < 1$ and $ab > 1$. From this we can construct a function that is interesting for our purposes, namely $\mathcal{W}_{d0}(x) \equiv x\mathcal{W}(x)$. I claim that $\mathcal{W}_{d0}(x)$ is differentiable at $x = 0$ but nowhere else. This is easy to show:

$$\lim_{h\to 0} \frac{(x+h)\mathcal{W}(x+h) - x\mathcal{W}(x)}{h} = \lim_{h\to 0}\left[x\frac{\mathcal{W}(x+h)-\mathcal{W}(x)}{h} + \mathcal{W}(x+h)\right]$$
$$= \begin{cases} \lim_{h\to 0}\mathcal{W}(h) = \mathcal{W}(0) & \text{for } x = 0 \\ \langle\,\nexists\,\rangle & \text{for } x \neq 0 \end{cases}, \tag{30}$$

where we have used the continuity of $\mathcal{W}(x)$ to show the line for $x = 0$ and employ an argument of reduction to a contradiction for $x \neq 0$. Suppose the limit for $x \neq 0$ exists. Since $\lim_{h\to 0}\mathcal{W}(x+h) = \mathcal{W}(x)$ exists and x is an h-independent factor, existence of the total limit would imply also the existence of $\lim_{h\to 0}(\mathcal{W}(x+h) - \mathcal{W}(x))/h$, which would mean that $\mathcal{W}(x)$ would be differentiable at x. But it was proven (by Hardy [1]) to be nondifferentiable.[22]

We now consider our two expressions.

$$\mathcal{E}_1[0] = \lim_{h\to 0}\lim_{x\to 0}\frac{\mathcal{W}_{d0}(x+h) - \mathcal{W}_{d0}(x)}{h} = \lim_{h\to 0}\frac{\mathcal{W}_{d0}(h)}{h} = \lim_{h\to 0}\frac{h\mathcal{W}(h)}{h} = \mathcal{W}(0)\,, \tag{31}$$
$$\mathcal{E}_2[0] = \lim_{x\to 0}\lim_{h\to 0}\frac{\mathcal{W}_{d0}(x+h) - \mathcal{W}_{d0}(x)}{h} = \langle\,\nexists\,\rangle\,. \tag{32}$$

In the formula for $\mathcal{E}_2[0]$, the inner limit does not exist (see Eq. (30)), because $x \neq 0$ there. Expression $\mathcal{E}_1[0]$ picks out the value of the derivative at $x = 0$ and therefore gives a result. Expression $\mathcal{E}_2$ tries to calculate a limit of a derivative function which cannot succeed, because the derivative exists only in one point.

Having understood the meaning of expressions $\mathcal{E}_1[x_0]$ and $\mathcal{E}_2[x_0]$ for *finite* x_0, let us now move on to the case $x_0 = \infty$. We start with $\mathcal{E}_2[\infty]$, which is easier to interpret,

$$\mathcal{E}_2[\infty] = \lim_{x\to\infty}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{x\to\infty} f'(x)\,, \tag{33}$$

where the second equality holds if $f(x)$ is differentiable at x. Then the inner limit exists and gives the derivative. For the whole expression to make sense, it is sufficient that there is some finite value a so that the inner limit exists for all $x > a$. Then we can check whether the outer limit exists using definition **D2**. If it exists and takes on some value d, then $\mathcal{E}_2[\infty] = d$ is the limit of the derivative of $f(x)$, as x approaches infinity. If it does not exist, i.e., $\lim_{x\to\infty} f'(x) = \langle\,\nexists\,\rangle$, then $\mathcal{E}_2[\infty] = \langle\,\nexists\,\rangle$, and we have shown that the derivative of $f(x)$ does not have a limit as x approaches infinity. This is the result that we obtain for $f(x) = \frac{\sin x^2}{x}$.

Now consider

$$\mathcal{E}_1[\infty] = \lim_{h\to 0}\lim_{x\to\infty}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\lim_{x\to\infty} f(x+h) - \lim_{x\to\infty} f(x)}{h} = \lim_{h\to 0} 0 = 0\,. \tag{34}$$

[22] For the limit as h approaches zero, x is a constant factor, which cannot make the difference quotient into which it is multiplied covergent – unless it is zero.

The *result* is simple here, but as we shall see, it does not have a meaningful interpretation expressible via derivatives of f within standard analysis. The second equality holds if $\lim_{x\to\infty} f(x)$ exists, and since in that case $\lim_{x\to\infty} f(x+h)$ must give the same value, the numerator reduces to zero and the outer limit becomes trivial. However, what is the meaning of the expression after the second equality? Because $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is not defined at infinity ($\{\infty\} \notin \mathbb{R}$), so we cannot interpret the numerator $\lim_{x\to\infty} f(x+h) - \lim_{x\to\infty} f(x)$ as part of a difference quotient anymore. Even if $f(\infty)$ were defined, we would not necessarily have $\lim_{x\to\infty} f(x) = f(\infty)$, because just as our example $v(x)$ above was defined at $x = 0$ but different from $\lim_{x\to 0} v(x)$, $\lim_{x\to\infty} f(x)$ may exist but be different from $f(\infty)$. Therefore, we have *no* valid interpretation for $\mathcal{E}_1[\infty]$ *in terms of the derivative of* f.

However, if we are willing to introduce a few elements of non-standard analysis, we can provide an interpretation that parallels that of the finite-x_0 case. I will discuss only the case of the projectively extended real line $\widehat{\mathbb{R}}$ here, assuming that the readers will themselves be able to construct the analog for the affinely extended real line $\overline{\mathbb{R}}$. In the case of $\widehat{\mathbb{R}}$, existence of $\lim_{x\to\infty} f(x)$ means that the limit must also exist, as x approaches $-\infty$, and that the two limits *must be the same*. Since this is true for $f(x) = \frac{\sin x^2}{x}$, we may use $\widehat{\mathbb{R}}$ and can avoid the restriction to one-sided derivatives necessary at $\pm\infty$ in $\overline{\mathbb{R}}$.

Let us then assume f to be a real function on $\widehat{\mathbb{R}}$, i.e., $f : \widehat{\mathbb{R}} \to \mathbb{R}$.[23] Then $f(\infty)$ should be defined. What value should we assign to $f(x) = \frac{\sin x^2}{x}$ at infinity? Well, it is natural to set $f(\infty) = \lim_{x\to\infty} f(x) = 0$.[24] Then the inner limit of $\mathcal{E}_1[\infty]$ can be written as $(\lim_{x\to\infty} f(x+h) - \lim_{x\to\infty} f(x))/h = (f(\infty+h) - f(\infty))/h$ and the outer limit produces $f'(\infty)$ by definition. So we obtain

$$\mathcal{E}_1[\infty] = \lim_{h\to 0} \frac{f(\infty+h) - f(\infty)}{h} = f'(\infty) = 0\,, \tag{35}$$

which means we have succeeded in calculating the derivative of f at infinity! Essentially, this is a *definition* of the derivative at infinity rather than a calculation. Then what is so interesting about this?

Well, here is the point that I find interesting. To define $f(\infty)$ we had to invoke the limit of $f(x)$ as x approaches infinity. That makes f continuous at infinity. An engineer might think this is not a big deal, because there is no other meaningful way to extend the definition of f to the domain $\widehat{\mathbb{R}}$. Then, on the one hand continuity at infinity seems to imply differentiability at infinity.[25] On the other hand, expression $\mathcal{E}_1[\infty]$ provides us with a definition of $f'(\infty)$ that makes $f'(x)$ *discontinuous* at infinity. After all, the full definition of the derivative on the whole domain is now (if we define $f(0)$ to be the limit as $x \to 0$ of $f(x)$)

$$\begin{aligned} &f' : \widehat{\mathbb{R}} \to \mathbb{R} \\ &f'(x) = \begin{cases} 2\cos x^2 - \frac{\sin x^2}{x^2} & \text{for } x \neq 0, x \neq \infty \\ 1 & \text{for } x = 0 \\ 0 & \text{for } x = \infty \end{cases} \quad . \end{aligned} \tag{36}$$

[23] We could also include ∞ in the codomain of f, i.e., assume $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$. But that would have no effect on the discussion.

[24] This is by no means compulsory. If we consider our real function a restriction from the corresponding complex function to the domain $\widehat{\mathbb{R}}$, then we could assign *any* value to $f(x)$, because $f(x) = \frac{\sin x^2}{x}$ has an essential singularity at the point infinity of the extended complex plane, meaning that $f(x)$ takes on almost every complex value in any neighborhood of $x = \infty$. However, taking $f(\infty) \neq 0$ would make the *real* function discontinuous at infinity, and we would again not be able to assign meaning to $\mathcal{E}_1[\infty]$ in terms of a derivative of f.

[25] To be checked further...

It is easy to verify that $f'(x)$ is continuous at $x = 0$, in fact for all $x \in \mathbb{R}$, but is discontinuous at $x = \infty$. The last statement follows from the consideration of $\mathcal{E}_2[\infty]$ presented above.

Therefore, we have the interesting situation that $f'(x)$ *can* be defined at $x = \infty$, but *not* by taking the limit of $f'(x)$ as $x \to \infty$, because that limit does not exist, rather via its definition in terms of $f(x)$ and its difference quotient. Then the engineer who thought that the only reasonable extension of a function to the domain $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ would be to set its value at infinity equal to its limit as its argument approaches infinity, would have to realize that this very reasonable approach to defining $f(\infty)$ leads to a definition of $f'(\infty)$ in an equally natural way that does not satisfy his constraint of reason, because $f'(x)$ is different at infinity from its limit at infinity, i.e., it is discontinuous there.

Nevertheless, this definition of $f'(\infty)$ looks quite consistent, because it leads to the result zero always, if f remains continuous and finite at infinity. Clearly, this is the correct result also in cases where $\lim_{x\to\infty} f'(x)$ exists, because then the conditions of theorem **P1** apply. This then corresponds to a situation where $f'(x)$ remains continuous at $x = \infty$ and therefore, the double limits $\mathcal{E}_1[\infty]$ and $\mathcal{E}_2[\infty]$ give the same result.

Let me summarize what we have learned. Barkana's "proof" that the limit of the derivative of $f(x)$, as x approaches ∞, must be zero, if $\lim_{x\to\infty} f(x) = \text{const.}$ does not work. The expression $\mathcal{E}_1$ that he uses to establish this result never calculates a limit of the derivative of $f(x)$. Instead, $\mathcal{E}_1[x_0]$ with x_0 finite calculates $f'(x_0)$, if $f(x)$ is continuous in a neighborhood of x_0; $\mathcal{E}_1[\infty]$ always is zero but cannot be interpreted as a derivative or limit of a derivative for functions defined on $\mathbb{R}$, it is the limit of a difference quotient that loses all information on functions $f(x)$ satisfying the aforementioned limit condition (being zero no matter what). If we extend the domain of functions to $\mathbb{R} \cup \{\infty\}$, then the interpretation of the inner limit of $\mathcal{E}_1$ as a difference quotient at a particular value of the argument x remains feasible and the outer limit then calculates $f'(\infty)$ which hence is zero. But again, no *limit* of $f'(x)$ is calculated, just a *value*. (There is no third limit that would act on a derivative, the other two are acting only on difference quotients.) Of course, the possibility of defining $f'(\infty)$ this way and the fact that it is zero under the conditions specified, is an interesting result in itself

On the other hand, the expression $\mathcal{E}_2$ does precisely what is needed for the answer to the original ResearchGate question. Its inner limit calculates $f'(x)$ and the outer limit is the limit of $f'(x)$, as x approaches x_0 in the finite case ($\mathcal{E}_2[x_0]$) and infinity in the case of $\mathcal{E}_2[\infty]$.

Reference

[1] G. H. Hardy: Weierstrass's non-differentiable function. In: *Trans. Amer. Math. Soc.* **17**, 301–325 (1916)
https://www.ams.org/journals/tran/1916-017-03/S0002-9947-1916-1501044-1/home.html