Sagnac effect and uniform speed of light

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There have been recent claims, in Research Gate question threads, that the Sagnac effect is not reconcilable with a uniform speed of light and that it requires a notion of absolute simultaneity. This would then suggest the existence of a preferred frame of reference, undermining the basic philosophy of special relativity. Such claims have been put forward, for example, by Stefano Quattrini on various occasions.

The purpose of this small essay is to refute these claims and, hopefully, to clarify things a bit, replacing prejudice with knowledge. While I have written a paper [1] touching on the topic more than 10 years ago, a less prosaic exposition may be useful.

Before discussing the effect itself, I would like to tell a little story. It will connect with the physics later.

When I was a postdoc, I worked at the Massachusetts Institute of Technology, for a little more than a year. So I had to go to Boston. Suppose, I wanted to visit this place again to have some nice scientific discussions.

Nowadays, I would fly from Berlin to Boston. The distance between the two cities is 6075 km. Looking up a possible flight, I find take-off to be possible at Berlin Brandenburg airport at 12:10 o'clock and arrival in Boston at 17:30. A return flight would by available a few days later with departure at 19:15 from Boston Logan International airport and touchdown in Berlin at 11:20 the following morning.

Of course, I am interested in how fast my airplane will go. For the outgoing flight, a quick calculation gives a time of flight of 5:20 hours, corresponding to an average velocity of 1139 km/h. That is fast! Not much below the speed of sound! Well, something must be wrong here, because the Concorde is not flying anymore, and other planes simply will not cruise at such speeds. Immediately, I recognize my error; Boston is in a different time zone from Berlin, its local time being six hours earlier. My velocity calculation was done with two different time coordinates *that were not synchronized*. To obtain the correct speed, I would have to add six hours to the nominal time difference between departure and arrival, giving a total duration of 11:20 h and an average velocity of the plane of only 536 km/h. Now, the pilot would agree...

Incidentally, I would also have obtained these 11:20 h as flight time by using my own watch to measure it, before readjusting it to the local time on arrival. Similarly, taking the local times of the cities to determine the duration of the return flight, I would obtain 16:05 h, corresponding to a measly average velocity of 378 km/h. Subtracting the 6 hours that I lost in switching time zones, the duration becomes 10:05 hours, and the average speed increases to 602 km/h.

I invite the readers to ponder which method of velocity determination is the correct one – putting the difference of local times in the denominator of the velocity expression or correcting (by addition or subtraction) for the difference between the local times in Berlin and Boston, resulting from a sum of (six one-hour) time gaps, appearing at the border of each of the time zones traversed.

The Sagnac effect

Let us now consider the Sagnac effec on a rotating disk. Say, the disk has a radius of R, the angular frequency of its rotation is ω, and we assume ω to be positive, corresponding to counterclockwise rotation. Light rays are sent around the disk, guided along its circumference, so they will travel at a distance R from its center.[1]

To actually measure the Sagnac effect, the two rays are made to interfere on return to the emitter and the phase shift of the interference pattern with respect to that for the same experiment on the stationary, i.e. non-rotating, disk is determined. We will consider the difference in time of flight of the two rays and not focus too much on ensuing phase shift, although that is also interesting. A *non-rotating* disk will give the *same* time of flight $2\pi R/c$ for the two light rays, and this defines the interference pattern with phase shift zero.

It is easy to calculate the times of flight in the laboratory frame, an inertial system, in which the disk center is at rest. The ray in the corotating direction will blast off, go around the disk and catch up with the moving emitter/observer. It has to cover a distance $2\pi R$ plus the distance by which the observer has moved during the entire trip of the ray. This gives rise to the following equation, from which the round-trip time t_+ can be evaluated:

$$ct_+ = 2\pi R + \omega R t_+ \qquad \Rightarrow \qquad t_+ = \frac{2\pi R}{c(1 - \omega R/c)} \tag{1}$$

The ray in the counterrotating direction will not have to cover the full circumference, as the emitter moves towards it on return, so it takes a smaller time:

$$ct_- = 2\pi R - \omega R t_- \qquad \Rightarrow \qquad t_- = \frac{2\pi R}{c(1 + \omega R/c)} \tag{2}$$

The time difference is

$$\Delta t = t_+ - t_- = \frac{2\pi R}{c}\frac{2\omega R/c}{1 - \omega^2 R^2/c^2} = \gamma^2 \frac{4\pi R^2 \omega}{c^2} = \gamma^2 \frac{4A\omega}{c^2}\,, \tag{3}$$

where $\gamma = \left(1 - \frac{\omega^2 R^2}{c^2}\right)^{-1/2}$ is the standard Lorentz factor (the velocity of the disk rim is $v = \omega R$) and A is the area of the disk. Since the difference in optical paths of the two rays is $c\Delta t$, the phase shift $\Delta\vartheta$ of the interference pattern is, given that the wavelength of the light is λ (assuming both rays to have the same frequency $\nu = c/\lambda$):

$$\Delta\vartheta = 2\pi \frac{c\Delta t}{\lambda} = \gamma^2 \frac{8\pi}{\lambda c} A\omega\,. \tag{4}$$

In the literature, you will often find that formula without the prefactor γ^2, because in practical applications of earth-bound gyroscopes based on the Sagnac effect, it is close to one ($v \ll c$).

These are the predictions of special relativity for what a laboratory observer would measure and they are borne out by experiment. Their derivation uses the independence of the speed of light of the source velocity plus Einstein synchronization for laboratory clocks.[2]

What are the predictions for observations by an observer sitting on the disk who is emitting the light rays? For easy reference, let us call her Dorothy. Well, the inertial observer at the disk center, whom we will call Inga, notes that the emitter moves at a velocity $v = \omega R$ with

[1] The refractive index at the center of the guiding fiber should be one, so we effectively have light traveling in vacuum. Of course, the effect is also present, if the light moves through a medium. There would only be very minor modifications to the discussion given here.

[2] Since the emission and reception of the light rays does not happen at the same positions in the lab – due to the rotation of the disk the emitter has moved to new positions at the times of reception – time measurements are not carried out with the same local clocks, so synchronization does play a role for the time intervals. The shift of the interference pattern is, however, independent of synchronization.

respect to herself at all times, so there should be time dilation by a factor $1/\gamma$. Hence, Dorothy should measure the following times:

$$\tau_+ = \frac{t_+}{\gamma} = \frac{2\pi R}{c(1-\omega R/c)}\left(1-\frac{\omega^2R^2}{c^2}\right)^{1/2} = \frac{2\pi R}{c}\left(\frac{1+\omega R/c}{1-\omega R/c}\right)^{1/2} , \tag{5}$$

$$\tau_- = \frac{t_-}{\gamma} = \frac{2\pi R}{c(1+\omega R/c)}\left(1-\frac{\omega^2R^2}{c^2}\right)^{1/2} = \frac{2\pi R}{c}\left(\frac{1-\omega R/c}{1+\omega R/c}\right)^{1/2} , \tag{6}$$

$$\Delta\tau = \tau_+ - \tau_- = \frac{t_+ - t_-}{\gamma} = \gamma\frac{4\pi R^2\omega}{c^2} , \tag{7}$$

where the last result can be simply read off from (3).[3]

To obtain the phase shift found by the disk observer Dorothy, we have to take into account that for her the wavelength of the light is different from λ. Due to her time being slowed down by a factor of $1/\gamma$ with respect to the proper time of the center observer, she must emit light of the frequency $\nu' = \gamma\nu$, in order for it to have the frequency ν for Inga. But then the wavelength in Dorothy's frame is $\lambda' = c/\nu' = \lambda/\gamma$ which leads to a phase shift of

$$\Delta\vartheta' = 2\pi\frac{c\Delta\tau}{\lambda'} = 2\pi\gamma\frac{4\pi R^2\omega}{c}\frac{\gamma}{\lambda} = \gamma^2\frac{8\pi^2R^2}{\lambda c}\omega = \gamma^2\frac{8\pi}{\lambda c}A\omega , \tag{8}$$

which is the same as Eq. (4). This was to be expected, as phase shifts are scalars, so they must be the same for all observers. So far, there are no problems with a special relativistic description of the effect.

Given the times of flight from Eqs. (5,6), we may calculate the average velocities $\bar{c}_\pm$ of the two light rays according to Dorothy. To do so, we need the path length covered by the light, which is the same in both directions. Its value is not $L = 2\pi R$ but $L' = 2\pi R\gamma$, due to the non-Euclidean geometry of the rotating disk [2].[4] Therefore, we obtain

$$\bar{c}_+ = \frac{L'}{\tau_+} = \gamma c\left(\frac{1+\omega R/c}{1-\omega R/c}\right)^{-1/2} = c\left[\frac{1-\omega R/c}{(1+\omega R/c)\,(1-\omega^2R^2/c^2)}\right]^{1/2} = \frac{c}{1+\omega R/c} , \tag{9}$$

$$\bar{c}_- = \frac{L'}{\tau_-} = \gamma c\left(\frac{1-\omega R/c}{1+\omega R/c}\right)^{-1/2} = c\left[\frac{1+\omega R/c}{(1-\omega R/c)\,(1-\omega^2R^2/c^2)}\right]^{1/2} = \frac{c}{1-\omega R/c} . \tag{10}$$

Detailed local considerations

Is this a problem for special relativity?[5] Not really. In an accelerated system, the speed of light can vary, and so the average speed along a loop may well be different from the speed at the position of a single observer on that loop. All that special relativity requires is that this observer will measure the speed of light to be c at her own position, if she uses standard rulers and standard clocks to do so.[6]

[3] The direct algebraic evaluation from the results for τ_+ and τ_- is not difficult either.

[4] A simple way to rationalize this is that if we cover the circumference with standard rulers (at rest with respect to the disk), the lengths of these have to add up to $2\pi R$ in the laboratory frame. But these are Lorentz contracted rods as they are moving with respect to the lab frame at velocity v and are oriented parallel to their velocity. So their rest length on the rotating disk is larger by a factor of γ which then must be true for the disk circumference as well. In footnote 12, an explicit calculation within the non-Euclidean geometry of the disk is given.

[5] And in particular, is it a problem that $\bar{c}_-$ exceeds the vacuum speed of light?

[6] If the one-way speed is measured, then an additional condition for the measurement to give c is that the clocks are Poincaré-Einstein synchronized, i.e., their synchronization must be equivalent to the one obtained by the procedure Einstein suggested in his 1905 paper on special relativity (and Poincaré somewhat earlier).

Indeed, if Dorothy chooses a comoving frame of reference, the axes of which remain parallel to an inertial frame, she will find the speed of light to vary along the circumference of the disk. In such a frame, every distant point of the disk would revolve about her position with an angular frequency $\gamma\omega$.[7] If the orientation of her line of sight is towards the center from the easternmost point of the disk at time τ_0, that orientation will be westward at τ_0, southward at time $\tau_0 + \pi/(2\gamma\omega)$, eastward at $\tau_0 + \pi/(\gamma\omega)$ and westward again at $\tau_0 + 2\pi/(\gamma\omega)$, as long as Dorothy does not move. She could realize this by spinning up a gyroscope at time τ_0 so that it points westward, parallel to her line of sight. At time $\tau_0 + \pi/(\gamma\omega)$, it would then be antiparallel to her line of sight.

While Dorothy would then measure c for the speed of light at her own position, she would "measure" values different from c at other points of the circumference of the disk. A "measurement" might consist of a measurement by a local observer, whose positions and times would have to be transformed to the frame of Dorothy, and since the transformations involve rotating coordinates, they would not be the Lorentz transformations. As a result, the speed of light, which would be c according to the local observer, would be different from c in Dorothy's frame. And it might well depend on the orientation of the light path with respect to the sense of rotation. We will not pursue this scenario any further, because the maths could be hairy. Rather let us be satisfied with the recognition that it is unlikely to uncover a problem with relativity, as the different coordinate velocities of light along the circumference might well lead to the averages $\bar{c}_\pm$ for a full loop.

Constructing a locally Einstein synchronized frame

There are nicer frames of reference for Dorothy. She may consider all fixed disk points as non-moving with respect to herself (meaning that her coordinate axes are "glued to" and will rotate along with, the disk). Looking at a fixed point on the disk then means that she is *not* rotating about herself but keeps not only her position but also her orientation.[8] All observers that are stationary like her at the same radius should see the same local geometry, i.e., a measurement of a piece of arc length by a distant observer can be immediately related to that of a similar local piece of arc by a simple symmetry operation (a rotation about the disk center). So local length measurements at radius R by a distant observer can be immediately taken as results for a "measurement at a distance" by our global observer (Dorothy). If it is possible for her to establish a common time with those local observers, *their* velocity measurements are *hers*. Establishing a common time means synchronizing clocks at distant places. If the time is to be measured locally with a standard clock, meaning that it is a local proper time, then it is clear that synchronization is not possible for different radial positions on the disk, as clocks at these will tick at different rates. They move at different velocities around the disk center, so their time dilation factors w.r.t. the center are different. But clocks at the *same* radius can be synchronized according to Inga, our center observer,[9] so there is hope for Dorothy that she can achieve a common time with all the observers on the disk that live at *her* radius (R).

A standard way to synchronize clocks (à la Einstein) is by *slow clock transport*. So Dorothy sends two cousins, Colleen and Cleo, counterclockwise and clockwise around the disk, each

[7] Hence, the transverse velocity of such a point is the larger the farther it is away from Dorothy's position.

[8] Getting rid of rotation this way is a purely kinematic approach. Physically speaking, rotation is as absolute in special relativity as in Newtonian mechanics. Even in general relativity, Mach's principle – i.e., motion is only defined relative to objects, motion with respect to empty space is meaningless – is not fully realized. There is no change of frame of reference by which a rotating black hole can be transformed into a non-rotating one.

[9] Clocks at different radii can also be synchronized, if we allow them to tick faster or more slowly than a standard clock, to compensate for time dilation effects. The GPS clocks on satellites are examples for such non-standard clocks.

carrying a high precision clock that has been set to coincide with Dorothy's clock at the beginning of the journey (at time τ_i). They both move very slowly, but at constant velocity $u(\ll c)$ and $-u$, respectively. All three clocks are constructed identically; moreover, the cousins will meet many other people on their journey (along $r = R$) that have identical clocks and are eager to set them to the time displayed by the clock of the respective cousin, anticipating the value of establishing a common time at least on their ring-shaped subset of the disk-world.

A number of notable events will happen on the two cousins' journey, which I will describe first and explain immediately afterwards. Initially, everything goes as anticipated and clocks along the sector already covered by each cousin will be synchronized with their own clocks. However, when the cousins meet, it will unexpectedly not be midway along the circumference, that is not at the angular position π (if we denote Dorothy's angular coordinate as zero and let the angle increase counterclockwise to 2π). Rather, Colleen will have covered an angle $\alpha < \pi$ and Cleo an angle $\beta = 2\pi - \alpha > \pi$. Their clocks will show different times, Colleen's reading $\tau_i + \alpha R\gamma/u$ and Cleo's $\tau_i + \beta R\gamma/u$, with Cleo's clock being ahead of Colleen's by a time interval $\Delta\tau_g = \Delta\tau/2$, where $\Delta\tau$ is given by Eq. (7). (From this information, the angles α and β can be calculated.[10]) Hence, the synchronization procedure did not succeed completely, because disk inhabitants on both sides of the angular position α will have clocks that differ by $\Delta\tau_g$ when compared directly across the separating line. The cousins conclude that it seems still possible to establish a *local* notion of simultaneity[11] across the time zone boundary by considering two events on different sides to be simultaneous, if the concerned clock on the β side shows a time that is by $\Delta\tau_g$ larger than that shown by the relevant clock on the α side.

They then continue their journey (without ever having changed their velocities), to complete the loops and report back to Dorothy. The next notable event is Cleo's arrival at Dorothy's position. Cleo's clock reads, as expected $\tau = \tau_f = \tau_i + 2\pi R\gamma/u$, which is what a trip of length $2\pi R\gamma$ at speed u should take. But to her surprise, Dorothy's clock shows a smaller time

$$\tau = \tau_f - \Delta\tau_g = \tau_i + \frac{2\pi R\gamma}{u} - \frac{2\pi R^2\gamma\omega}{c^2} = \tau_i + 2\pi R\gamma\frac{1 - u\omega R/c^2}{u}\,, \tag{11}$$

corresponding to an average velocity $u/(1 - u\omega R/c^2)$. Finally, Colleen arrives at Dorothy's position. Her own clock also shows $\tau = \tau_f$ on arrival, but Dorothy's clock now shows a larger time

$$\tau = \tau_f + \Delta\tau_g = \tau_i + \frac{2\pi R\gamma}{u} + \frac{2\pi R^2\gamma\omega}{c^2} = \tau_i + 2\pi R\gamma\frac{1 + u\omega R/c^2}{u}\,, \tag{12}$$

corresponding to an average velocity $u/(1 + u\omega R/c^2)$. (And Cleo's clock displays $\tau = \tau_f + 2\Delta\tau_g = \tau_f + \Delta\tau$, assuming she has stayed with Dorothy and not readjusted her clock.) It may also be interesting to note that Cleo's or Colleen's "two-way velocity" u_{tw}, if either of them decided to retrace their path in the opposite direction (which means that Cleo would follow up her clockwise path with a counterclockwise one mimicking Colleen's original trip and vice versa for Colleen) would be given by

$$\frac{2}{u_{tw}} \equiv \frac{1 - u\omega R/c^2}{u} + \frac{1 + u\omega R/c^2}{u} = \frac{2}{u}\,, \tag{13}$$

i.e., it would just be u.

To explain these facts in the rotating frame directly, one has to do special relativity in an accelerated frame of reference, which means working with the appropriate metric. Some would

[10]The result is $\alpha = \pi(1 - uR\omega/c^2)$, $\beta = \pi(1 + uR\omega/c^2)$.

[11]For the definition of local velocities we do not need more than a local notion of simultaneity, because in the limit of an instantaneous velocity, the two times as well as positions needed to evaluate the defining difference quotient are to be taken arbitrarily close to each other.

consider this doing "general relativity", because the maths were developed during the construction of the generalized theory. However, the modern view is that special relativity is the theory of flat spacetime, regardless what mathematical tools we use in the description, while general relativity deals, beyond comprising special relativity, with the additional phenomena due to spacetime curvature. Hence, we are doing special relativity when we use metric tensors to describe accelerated systems. Nevertheless, I will relegate the metric description to a footnote that readers not familiar (or willing to deal) with the manipulation of line elements may skip.[12] In the main text, I will simply explain what happens from the point of view of our inertial observer Inga. Most people will find that physically more transparent.

First, we note that for Inga the velocities of Colleen and Cleo are given by

$$u_{\mathrm{Co}} = \frac{u+v}{1+uv/c^2} = \frac{u+\omega R}{1+u\omega R/c^2}\,, \tag{14}$$

$$-u_{\mathrm{Cl}} = \frac{-u+v}{1-uv/c^2} = \frac{-u+\omega R}{1-u\omega R/c^2}\,, \tag{15}$$

with a little help from the relativistic velocity addition theorem. This immediately allows to calculate the round trip times of the cousins:

$$u_{\mathrm{Co}}\, t^+_{\mathrm{Co}} = 2\pi R + v\, t^+_{\mathrm{Co}} \qquad \Rightarrow \qquad t^+_{\mathrm{Co}} = \frac{2\pi R}{u_{\mathrm{Co}} - v}\,, \tag{16}$$

$$u_{\mathrm{Cl}}\, t^-_{\mathrm{Cl}} = 2\pi R - v\, t^-_{\mathrm{Cl}} \qquad \Rightarrow \qquad t^-_{\mathrm{Cl}} = \frac{2\pi R}{u_{\mathrm{Cl}} + v}\,, \tag{17}$$

and the velocity differences in the denominators are given by

$$u_{\mathrm{Co}} - v = \frac{u+v}{1+uv/c^2} - v = u\frac{1-v^2/c^2}{1+uv/c^2} = \frac{u}{\gamma^2}\frac{1}{1+uv/c^2}\,, \tag{18}$$

$$u_{\mathrm{Cl}} + v = \frac{u-v}{1-uv/c^2} + v = u\frac{1-v^2/c^2}{1-uv/c^2} = \frac{u}{\gamma^2}\frac{1}{1-uv/c^2}\,, \tag{19}$$

[12] Written in cylindrical coordinates, the Minkowski line element takes the form: $\mathrm{d}s^2 = -c^2\mathrm{d}t_I^2 + \mathrm{d}r^2 + r^2\mathrm{d}\varphi_I^2 + \mathrm{d}z^2$, where the subscript I refers to the fact, that t_I is a proper time for Inga and the angular coordinate φ_I measured with respect to her nonrotating coordinate system. φ_I will typically vary (linearly) in time for fixed points on the rotating disk. To obtain a disk stationary coordinate frame, set $\varphi = \varphi_I - \omega t_I$, i.e. $\mathrm{d}\varphi_I = \mathrm{d}\varphi + \omega \mathrm{d}t_I$, which produces a metric that is not time orthogonal. We have: $\mathrm{d}s^2 = -\left(1-\omega^2r^2/c^2\right)c^2\mathrm{d}t_I^2 + 2\omega r^2\mathrm{d}t_I\mathrm{d}\varphi + \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2$. The presence of the mixed term $\propto \mathrm{d}t_I\mathrm{d}\varphi$ shows that the time coordinate t_I is not orthogonal to all spatial coordinates, which means that if we set the line element equal to zero, which describes the propagation of light, the resulting quadratic equation for $\mathrm{d}t_I$ will have two significantly different solutions rather than one solution and its negative; this means that light takes different times in the forward and backward directions along the spatial element $(0, r\mathrm{d}\varphi, 0)$. The metric can be made time orthogonal via completion of squares: $\mathrm{d}s^2 = -\left(1-\omega^2r^2/c^2\right)c^2\left(\mathrm{d}t_I - ((\omega r^2/c^2)/(1-\omega^2r^2/c^2))\,\mathrm{d}\varphi\right)^2 + \mathrm{d}r^2 + (1/(1-\omega^2r^2/c^2))r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2$. This shows that $\mathrm{d}\tilde{t} = \left(1-\omega^2r^2/c^2\right)^{1/2}\left(\mathrm{d}t_I - ((\omega r^2/c^2)/(1-\omega^2r^2/c^2))\,\mathrm{d}\varphi\right)$ is a time differential that is orthogonal to all spatial coordinates. It is not a total differential, however, so integrating it from one spacetime event to another along different paths may give different results. Nevertheless, its integral may serve as a "local time coordinate" (Langevin). From the $\mathrm{d}\varphi$ term of the line element, we can find the circumference of the disk, demonstrating the non-Euclidean nature of the geometry: $L' = \int_0^{2\pi} r/(1-\omega^2r^2/c^2)^{1/2}\big|_{r=R}\mathrm{d}\varphi = 2\pi R\gamma$. Dorothy's proper time increment is given by $\mathrm{d}\tau_D = \left(1-\omega^2r^2/c^2\right)^{1/2}\mathrm{d}t_I$, which allows us to rewrite the differential of the local time at $r = R$ as $\mathrm{d}\tilde{t} = \mathrm{d}\tau_D - \gamma(\omega R^2/c^2)\,\mathrm{d}\varphi$. The time $\tilde{t}$, which runs at the same rate as Dorothy's proper time τ_D for circumference stationary observers, is the time that the cousins are trying to establish by Einstein synchronization via slow clock transport (because with that time the metric becomes time orthogonal). Their speed is $u = \pm\gamma R\mathrm{d}\varphi/\mathrm{d}\tilde{t}$, which allows us to express $\mathrm{d}\tilde{t}$ by $\mathrm{d}\varphi$ along their trajectories and hence to determine the time interval that passes during one trip around the disk. Along Colleen's trajectory, $\mathrm{d}\tilde{t} = (\gamma R/u)\mathrm{d}\varphi$, hence $\mathrm{d}\tau_D = (\gamma R/u)\mathrm{d}\varphi + \gamma(\omega R^2/c^2)\mathrm{d}\varphi$. Integrating over the full circle, we get $\tau_f^+ - \tau_i = 2\pi R\gamma(1+u\omega R/c^2)/u$. For Cleo, who arrives earlier, we have $\mathrm{d}\tilde{t} = -(\gamma R/u)\mathrm{d}\varphi$, hence $\mathrm{d}\tau_D = -(\gamma R/u)\mathrm{d}\varphi + \gamma(\omega R^2/c^2)\mathrm{d}\varphi$. Integration is now from 2π to zero, and we obtain $\tau_f^- - \tau_i = 2\pi R\gamma(1-u\omega R/c^2)/u$. The difference $\tau_f^+ - \tau_f^- = 4\pi\gamma R^2\omega/c^2 = \Delta\tau$ is *independent* of u.

hence

$$t^+_{\mathrm{Co}} = \frac{2\pi R\gamma^2}{u}\left(1+\frac{u\omega R}{c^2}\right) , \tag{20}$$

$$t^-_{\mathrm{Cl}} = \frac{2\pi R\gamma^2}{u}\left(1-\frac{u\omega R}{c^2}\right) . \tag{21}$$

Multiplying this with the time dilation factor $1/\gamma$ that applies to stationary disk observers (and adding the start time τ_i), we recover (12) and (11). Since the velocity differences (18,19) describe, how fast Colleen and Cleo move away from Dorothy, it is obvious that their meeting point must be at an angle that divides the full angle of 2π in the proportion corresponding to their ratio:

$$\frac{\alpha}{2\pi-\alpha} = \frac{u_{\mathrm{Co}}-v}{u_{\mathrm{Cl}}+v} = \frac{1-uv/c^2}{1+uv/c^2} \qquad \Rightarrow \qquad \alpha = \pi\left(1-\frac{uv}{c^2}\right) , \tag{22}$$

and here no corrections are necessary to transform the result to the disk frame.

Why do the clocks of Colleen and Cleo differ from each other and from that of Dorothy, despite the explicit intent to synchronize clocks? From Inga's point of view that is pretty simple: all three of them move at different velocities, so their clocks must tick at different rates, as the time dilation factor relative to Inga will be different for each of them. In the following, I will denote by γ, as before, the Lorentz factor for the velocity $v=\omega R$. If a factor referring to a different velocity is to be given, I will write $\gamma(u)$, etc., i.e. the argument indicates the relevant velocity. We have

$$\begin{aligned}\gamma(u_{\mathrm{Co}}) &= \left(1-\frac{(u+v)^2}{\left(1+uv/c^2\right)^2c^2}\right)^{-1/2} = \left(\frac{c^4+\cancel{2uvc^2}+u^2v^2-u^2c^2-\cancel{2uvc^2}-v^2c^2}{c^4+2uvc^2+u^2v^2}\right)^{-1/2} \\ &= \left(\frac{(c^2-u^2)(c^2-v^2)}{(c^2+uv)^2}\right)^{-1/2} = \left(1+\frac{uv}{c^2}\right)\gamma(u)\gamma(v) , \end{aligned} \tag{23}$$

$$\begin{aligned}\gamma(u_{\mathrm{Cl}}) &= \left(1-\frac{(u-v)^2}{\left(1-uv/c^2\right)^2c^2}\right)^{-1/2} \\ &= \left(\frac{(c^2-u^2)(c^2-v^2)}{(c^2-uv)^2}\right)^{-1/2} = \left(1-\frac{uv}{c^2}\right)\gamma(u)\gamma(v) . \end{aligned} \tag{24}$$

Knowing the time dilation factors for Colleen and Cleo, we may use Eqs. (20,21) to obtain the amount of time that has passed according to their clocks, until they rejoin Dorothy:

$$\begin{aligned}\tau^+_{\mathrm{Co}} &= \frac{1}{\gamma(u_{\mathrm{Co}})}\frac{2\pi R\gamma^2}{u}\left(1+\frac{u\omega R}{c^2}\right) = \frac{1}{1+u\omega R/c^2}\frac{1}{\gamma(u)\gamma(v)}\frac{2\pi R\gamma(v)^2}{u}\left(1+\frac{u\omega R}{c^2}\right) \\ &= \frac{2\pi R\gamma(v)}{\gamma(u)u} \approx \frac{2\pi R\gamma}{u} = \tau_f-\tau_i , \end{aligned} \tag{25}$$

$$\begin{aligned}\tau^-_{\mathrm{Cl}} &= \frac{1}{\gamma(u_{\mathrm{Cl}})}\frac{2\pi R\gamma^2}{u}\left(1-\frac{u\omega R}{c^2}\right) = \frac{1}{1-u\omega R/c^2}\frac{1}{\gamma(u)\gamma(v)}\frac{2\pi R\gamma(v)^2}{u}\left(1-\frac{u\omega R}{c^2}\right) \\ &= \frac{2\pi R\gamma(v)}{\gamma(u)u} \approx \frac{2\pi R\gamma}{u} = \tau_f-\tau_i . \end{aligned} \tag{26}$$

Note that the results before the $\approx$ sign are exact, including the time dilation factor $1/\gamma(u)$ by which the cousins' clocks are slowed down with respect to Dorothy's. But if their journey has to serve the purpose of clock synchronization for clocks at rest on the disk rim, their speed u must be so small that $\gamma(u)$ can be replaced by one.

Finally, to obtain the time difference between the cousins' clocks at their meeting point, we simply have to replace the angle 2π in Eq. (25) by α and in Eq. (26) by $2\pi - \alpha$:

$$\begin{aligned}\tau^-_{\rm Cl}(2\pi-\alpha) - \tau^+_{\rm Co}(\alpha) &= \frac{(2\pi-\alpha)R\gamma}{u} - \frac{\alpha R\gamma}{u} = \frac{(2\pi-2\alpha)R\gamma}{u} \underset{(22)}{=} \frac{2\pi u v R\gamma}{c^2 u} \\ &= \frac{2\pi R^2\gamma\omega}{c^2} = \mathit{\Delta}\tau_g\,. \end{aligned} \tag{27}$$

Simultaneity issues

So what has been achieved by the attempt to establish Einstein synchronization along the disk circumference? A time coordinate has been introduced that has a discontinuity at one point ($\varphi = \alpha$) of the rim. We could consider this a *common time* between Dorothy and all the stationary circumference observers. Using this time in the denominator of a velocity definition, we would get a coordinate velocity. However, coordinate velocities obtained from different coordinates are not easy to compare; moreover, the second postulate of special relativity does not apply to arbitrary coordinate velocities. Rather, when developing special relativity, Einstein always tried to use spatial coordinates that directly translated into measurable distances and temporal coordinates that directly translated into measurable time differences, in short, he used inertial coordinates. Only later, when he generalized the theory to turn it into a theory of gravitation, became it necessary to admit arbitrary coordinates, which required the use of a translation tool from these coordinates to measurable lengths and time intervals, which is the metric. Since the definition of a "measurable" velocity requires, in general, the difference of two time readings (at different positions) in its denominator which must belong to a common time to define a meaningful time interval, we need an operational definition of common time rather than a purely coordinate based one (which need not care about discontinuities). What we require is that the time at the two positions runs at the same rate and that it takes the same value for two events that are simultaneous. Therefore, we need a definition of simultaneity.[13]

What should such a definition be based on? Well, a *causal theory of time* [3] would, after having defined causal connectibility of events,[14] establish as a necessary condition for simultaneity of two events that, if they are not identical,[15] they must not be causally connectible. For Newtonian mechanics, in which *any* two events are causally connectible, because gravitational influences travel at infinite speed, one could modify this as follows: first define a *before* and an *after* relation, saying that A is before B, if A can causally influence B but B cannot causally influence A, and that A is after B, if B can causally influence A but A cannot causally influence B. Then, a necessary condition for simultaneity of A and B would be that A is neither before nor after B. In Newtonian mechanics, this is sufficient to define an equivalence relation, in special relativity, it only means that A and B are spacelike, which is not an equivalence relation.[16] So a second requirement for simultaneity would be that it constitutes an equivalence relation, i.e., any event A is simultaneous with itself; if A is simultaneous with B, so is B with A, and if A is simultaneous with B and B simultaneous with C, then A is simultaneous with C. It should then immediately be clear that if we can construct a time coordinate so that two non-identical events having the same time coordinate are always spacelike, this is a valid simultaneity relation, because the fact of two events having the same time coordinate

[13] And one that can be used before we have an explicit time coordinate.

[14] Events A and B are causally connectible, if A could in principle exert a causal influence on B or vice versa.

[15] Meaning they do not correspond to the same point of spacetime.

[16] It is easy to construct examples of events A, B, and C, where A and B as well as B and C are spacelike, but A and C are timelike, hence the relation *spacelike* is not transitive.

establishes an equivalence relation.[17] Since it is possible, in general, to construct many such time coordinates, simultaneity based on a causal theory of time is not uniquely defined, there is some choice, meaning that simultaneity at a distance is *conventional*.

There was an influential paper in 1977 by Malament [4], in which he showed that the only equivalence relation between events on the world line O of an inertial observer and events outside of O that was *definable* using only O and the causal structure of spacetime, is the standard simultaneity relation, called Sim_O, obtained by Einstein synchronization. This was hailed as a milestone by many, in particular in the philosophical community, who believed the controversy about the conventionality of simultaneity decided in favor of the anti-conventionalists.[18] On the other hand, the physical community remained largely skeptical, because Malament turned the obvious non-uniqueness of simultaneity relations compatible with the causal structure of spacetime, allowing the requirement of additional conditions in order to pick out one of the many possible simultaneity relations, into a criterion for excluding most of them by the requirement that *no* additional condition in the form of parameters describing the respective simultaneity relation must be added. He couched this in terms of requiring simultaneity to be given by an equivalence relation that is invariant under causal automorphisms leaving the worldline O invariant. The only relationship that satisfies this is Sim_O.

Unfortunately, this construction works for straight world lines only, i.e., in inertial systems. In geometrical language, it essentially says that the only foliation of spacetime (giving as equivalence classes 3D sheets of constant time) that is definable using only the world line and the causal structure given by light cones consists of the hyperplanes orthogonal to the world line.[19] However, if the world line is not straight (and the world lines of stationary disk observers aren't, they are helices), then orthogonal hyperplanes on it will not divide the spacetime into equivalence classes, rather they will intersect each other. That means that if Malament's argument had decided the question in favor of Einstein synchronization for inertial systems, it would also lead to the conclusion that no simultaneity is possible at all in rotating systems. As I said before, the physics community remained largely unconvinced: why should one require a simultaneity relation to be invariant under the class of causal automorphisms that leave a given inertial world line invariant?

Getting back to the time coordinate that was established by the cousins, the fact that it has a discontinuity at one point shows that it cannot satisfy the conditions that we have for a simultaneity relation *globally*, i.e., around the full circumference. This is easy to see: if we send a light signal from the β side of the gap, where the time shown by clocks is later than on the α side, towards the latter, then that light ray will first cover a region where the time is smaller than at its starting point and will, as time passes, eventually arrive at a position where it has exactly the same value as at the starting point, thus connecting two positions with the same time on local clocks, which means these two events cannot be spacelike. Therefore, across the gap equal time coordinates do not mean simultaneity. The situation would be very much like in the story I told in the introduction: if my plane to Boston really had gone at a little more than 1000 km/h, so that it arrived *exactly* after six hours time of flight, the local time on arrival would be the same as the local time at departure, but arrival and departure would definitely not be simultaneous. The time discontinuity on the disk behaves, in all essential

[17] A has the same time coordinate as itself; if A and B have the same time coordinate, then B and A have the same time coordinate; if A and B have the same time coordinate and B and C have the same time coordinate, then A and C have the same time coordinate.

[18] But in a way that Quattrini who himself is an anti-conventionalist would not like: Malament's proof also excludes Quattrini's favorite "absolute" synchronization.

[19] That is so because orthogonality, i.e., a right angle, between a line and a hypersurface of codimension one is definable without any additional parameters, whereas for any other angle we need direction cosines or other trigonometic parameters.

aspects, exactly like the border of a time zone. So working with such a time coordinate does not pose a major problem. We know how to do it, as we do it on a regular basis on our planet. It just cannot serve as a globally synchronous time coordinate.

However, locally, i.e., for sufficiently short segments on the circumference, this time is Einstein synchronized by construction, that is close-by points, not separated by the discontinuity on the shortest arc connecting them along the rim, have a definite notion of simultaneity: if their synchronized clocks read the same time, the events of these readings are simultaneous.[20] Hence, we can meaningfully define a velocity measurable with standard rulers and standard clocks, using arc lengths along the circumference and the Einstein synchronized clocks of local observers everywhere except at the discontinuity point. Moreover, *at* this point, we can *also* define operational velocities by subtracting, for signals moving from the α to the β sector of the disk, the time gap $\Delta\tau_g$ from local clock readings there, and by adding, for signals moving from the β to the α sector, the time gap to clock readings in the latter. That this is the correct procedure, is obvious from the fact that the discontinuity of the time gap could have been shifted from its position at $\varphi = \alpha$ to larger values, if Colleen had insisted, after meeting with Cleo, that people in the β sector that had already synchronized their clocks with Cleo's, would now reset them to the reading on Colleen's clock. Which would mean that they would have to turn their clocks back by an amount Δt_g. Conversely, if Cleo had instead insisted that people on the α side of the boundary would reset their clocks to the setting of her own one, meaning they would set them ahead by Δt_g, this would have shifted the discontinuity towards smaller φ values.

Using the frame

Let us discuss a number of local velocity measurements by Dorothy based on this local Einstein simultaneity. To perform such a measurement, local clocks synchronized before by the passage of the cousins, should now be everywhere along the circumference, dividing it into many short segments. Each segment is occupied by a local observer who notes, when an object or a signal to be studied enters their segment, the time on the clock at the entry point, and when it leaves the segment, the time on the clock at the exit point. The observers then send the information about the length of their supervised segment as well as about the time interval the object/signal took between entry and exit, to Dorothy, who may then calculate the local velocity. In general, the time interval they send will be the difference between the exit and entry times of the object/signal; however, being intelligent observers, the one in whose segment the time gap is located (i.e., whose segment is cut by the angular coordinate $\varphi = \alpha$) will calculate the time needed by the signal by adding Δt_g to that time difference, if the signal was crossing the discontinuity point in the clockwise direction, and subtracting Δt_g, if the signal was crossing the discontinuity in the counterclockwise direction.[21] That is, she would apply the same kind of procedure that I would use on my flights to and from Boston, in order to obtain a realistic velocity of my plane.

What velocities would Dorothy measure for the journey of the two cousins? For the first part of Colleen's journey, the time intervals reported would be the same as those on Colleen's clock, obviously. The arc length intervals would also be the same, because all of them are measured

[20] That the condition of two events of the same time being spacelike is satisfied (in addition to the equivalence relation property guaranteed by the existence of a time coordinate), can be easily established for infinitesimally separated events from the metric description given in footnote 12.

[21] The local observers could also send the raw data, i.e., the arc length positions of the entry and exit points plus the time readings of the respective clocks. In that case, the observer on the segment containing the "border of the time zone" would send the information on that fact as well and on the size of the time gap to be added or subtracted to the relevant time interval. The calculations would then all be done by Dorothy.

using standard rulers, so any position passed by Colleen could already have been marked by an arc length coordinate even before she started her journey. During the part of Colleen's journey that is in the β region, the times read off by the local observers would be Δt_g larger than the time on Colleen's own clock, but the time intervals would again be the same as for Colleen. In the segment with the discontinuity, the entry time would be the same as on Colleen's clock, the exit time would be Δt_g larger, but the local observer would subtract the time gap, so she would report precisely the time interval that passed on Colleen's clock in the segment. Dorothy would find Colleen's velocity to be the same in all segments and to be equal to u. Moreover, she could sum up all the time intervals reported to calculate an average velocity for Colleen and would find that sum to be smaller by $\Delta \tau_g$ than the round trip time she measured with ther own clock (Eq. (12)), because of the subtraction of that time from one (and only one) of the reported time intervals. So she finds that the time *actually* taken by Colleen[22] is

$$2\pi R\gamma \frac{1+u\omega R/c^2}{u} - \frac{2\pi R^2\gamma\omega}{c^2} = \frac{2\pi\gamma R}{u}\,, \tag{28}$$

which makes the actual average velocity of Colleen equal to u, as it should be, given that her momentary velocity was u during the whole journey. Dorothy also realizes the reason why she got a different average velocity by simply using the time that passed on her own clock during Colleen's trip. While this is her proper time, therefore a time as real as any time can be, it is not the proper time of all observers along Colleen's path.[23] But by the synchronization procedure it has been extended to a valid coordinate time along Colleen's path (with the small disadvantage of having a discontinuity). So the average velocity she measured is a valid coordinate average velocity. The coordinate velocity, of which it is the average, is obtained by *not* making the correction for the gap at the segment containing the angle α. That is, the coordinate velocity of Colleen will be u everywhere except in that segment, where it will be reduced by a factor of $\Delta\tau_{\rm seg}/(\Delta\tau_{\rm seg}+\Delta\tau_g)$, $\Delta\tau_{\rm seg}$ being the time actually needed by Colleen to traverse the segment. This then leads to an overall average velocity of $u/(1+u\omega R/c^2)$.

Obviously, a very similar discussion can be given for the velocity of Cleo, with the time gap $\Delta\tau_g$ to be added by the local observer, in whose segment she crosses the discontinuity, leading to a local velocity of u in the clockwise direction, whereas the average coordinate velocity without that correction becomes $u/(1-u\omega R/c^2)$.

Since the time gap is independent of u, the *same* relationships between true local velocities and average coordinate velocities will be observed for observers or particles that move in counterclockwise and clockwise directions at a *large* velocity u. The only difference is that these will not measure the same time intervals as local disk observers with their carry-along clocks, because the latter will be retarded by time dilation. Slow clock transport is peculiar in that a one-way velocity can be measured with a single clock, taken from the first to the second position at which times are measured. This is possible, because we have a theory that tells us that the clock rate will not be affected by the transport, if it is slow enough.

After these lengthy preliminaries, we are now in a position to discuss local vacuum light speeds on the disk circumference and their relationship to the global average measured by Dorothy. First, it should be clear that due to the local Einstein synchronization, these light speeds must be the same in the counterrotating and the corotating directions. To obtain the actual value, I will not invoke Einstein's second postulate, although arguments can be easily advanced why it must hold locally in the accelerated system, too. Rather, I will invoke rotational symmetry: the local speed of light should be the same for all disk observers on the circumference. Let us assume its value to be w. Then we know from our considerations so far

[22] In both Colleen's and the circumference observers' view.

[23] I.e., it is not synchronous with all of those.

that the average speed measured by Dorothy for a full circle of a signal moving at speed w must be $\bar{w}_+ = w/(1 + w\omega R/c^2)$ for counterclockwise and $\bar{w}_- = w/(1 - w\omega R/c^2)$ for clockwise motion. We may then calculate the total time of flight around the disk, which must be equal to the one given by Eq. (5) or (6), respectively. Then we can solve for w, and a countercheck for the correctness of our result would be that we must get the same value for w from both equations. Let us see:

$$\begin{aligned}
&\frac{2\pi R\gamma}{w}\left(1 + \frac{w\omega R}{c^2}\right) = \tau_+ = \frac{2\pi R}{c}\left(\frac{1+\omega R/c}{1-\omega R/c}\right)^{1/2} = \frac{2\pi R\gamma}{c}\left(1 + \frac{\omega R}{c}\right) \\
&\quad\Rightarrow \qquad \frac{1}{w} + \frac{\omega \cancel{R}}{\cancel{c^2}} = \frac{1}{c} + \frac{\omega \cancel{R}}{\cancel{c^2}} \qquad \Rightarrow \qquad w = c\,, \qquad (29) \\
&\frac{2\pi R\gamma}{w}\left(1 - \frac{w\omega R}{c^2}\right) = \tau_- = \frac{2\pi R}{c}\left(\frac{1-\omega R/c}{1+\omega R/c}\right)^{1/2} = \frac{2\pi R\gamma}{c}\left(1 - \frac{\omega R}{c}\right) \\
&\quad\Rightarrow \qquad \frac{1}{w} - \frac{\omega \cancel{R}}{\cancel{c^2}} = \frac{1}{c} - \frac{\omega \cancel{R}}{\cancel{c^2}} \qquad \Rightarrow \qquad w = c\,. \qquad (30)
\end{aligned}$$

Hence, we conclude that the only value of the local speed of light compatible with Dorothy's measured average coordinate speed of light in either direction is the vacuum speed of light!

To summarize, we have seen that a local description of physics in the accelerating (non-Euclidean) frame of reference set up by stationary disk observers[24] can explain the Sagnac effect based on a locally Einstein synchronized time, and that such a description requires the local vacuum speed of light to be c everywhere on the disk circumference. This is in stark contrast to the claim by Quattrini and others that such an explanation requires absolute synchronization and that the local speed of light could not be c, as it had to be different in the counter- and corotating directions.

I will briefly discuss the description of the Sagnac effect in terms of a *centrally* synchronized time, which agrees with the so-called absolute synchronization, if the disk center happens to be at rest in the absolute frame.[25]

Central synchronization

What we discussed in the last section, was an operational procedure that provided us with a locally Einstein synchronized time coordinate on the disk circumference allowing us to give a flawless description of the Sagnac effect with a universal one-way speed of light for disk observers on the circumference. The time coordinate was *not* globally synchronized, due to the appearance of the time gap $\varDelta\tau_g$.

However, given this time coordinate, it is easy to construct a new one that is even globally synchronized (the synchrony is just not Einstein). Here is how. Our time coordinate τ, taken as a function of the azimuthal angle φ on the disk, has the property $\tau(\alpha^+) = \tau(\alpha^-) + \varDelta\tau_g$, where α^- is an angle that is infinitesimally smaller than α and α^+ is one that is infinitesimally larger. Otherwise, $\tau(\varphi)$ is continuous. Now let us require our circumference observers to reset

[24] A coordinate independent notion of a frame of reference defines it as a timelike congruence of world lines (of test particles or observers). The world lines of our stationary disk observers constitute such a congruence. Reference [5] explains how then a local splitting into time and a local space platform can be made and admissible coordinates may be introduced.

[25] I give it a different name as it is different from absolute synchronization whenever the disk center is at rest in a frame of reference that is moving with respect to the "absolute frame". In that case, absolute synchronization would not lead to a constant speed of light in either direction on the circumference of the disk.

their clocks to

$$\tilde{\tau} = \begin{cases} \tau + \varphi \Delta\tau_g/(2\pi) & \text{for } \varphi < \alpha \\ \tau + (\varphi - 2\pi)\Delta\tau_g/(2\pi) & \text{for } \varphi > \alpha \end{cases}, \tag{31}$$

assuming they all know their own angular positions. Then we obviously have

$$\begin{aligned} \tilde{\tau}(\alpha^-) &= \tau(\alpha^-) + \alpha\Delta\tau_g/(2\pi) = \tau(\alpha^+) - \Delta\tau_g + \alpha\Delta\tau_g/(2\pi) \\ &= \tau(\alpha^+) + (\alpha - 2\pi)\Delta\tau_g/(2\pi) = \tilde{\tau}(\alpha^+)\,, \end{aligned} \tag{32}$$

meaning that the time coordinate $\tilde{\tau}$ is continuous on the whole circumference (and if there is an observer at $\varphi = \alpha$, she should set her clock to $\tilde{\tau}(\alpha^-) = \tilde{\tau}(\alpha^+)$). Moreover, it is easy to show that equality of this time coordinate for two different events means they are spacelike. Therefore, $\tilde{\tau}$ is a time coordinate that produces a globally valid simultaneity relation on the disk circumference. In fact, this synchrony can be operationally established without the detour via a locally Einstein synchronized time simply by Inga sending a light signal from the disk center to all circumference observers, on reception of which they all set their clocks to a predefined time value τ_p. If the goal rather is to synchronize all clocks with that of Dorothy without changing hers, that can be achieved by a two-step procedure. First, all circumference observers except for Dorothy set their clocks to τ_p on reception of Inga's signal. Dorothy calculates the offset $\tau - \tau_p$ between her local time and the predefined time and broadcasts the result to all the other observers, who then advance their clocks by $\tau - \tau_p$. After the – finite – time it takes for all observers to have received Dorothy's message (plus a small delay), all clocks will be centrally synchronized with Dorothy's.[26]

Let us calculate the velocity of light in terms of the central synchronization. For the Einstein synchronized case, we have $\mathrm{d}s/\mathrm{d}\tau = \pm c$ for counterclockwise and clockwise motion, respectively. $\mathrm{d}s$ is shorthand for the arc length element $R\gamma\mathrm{d}\varphi$. With central synchronization, the velocity then becomes:

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}\tilde{\tau}} &= \frac{\mathrm{d}s}{\mathrm{d}\tau + \Delta\tau_g \mathrm{d}\varphi/(2\pi)} = \frac{\mathrm{d}s}{\mathrm{d}\tau + \Delta\tau_g \mathrm{d}s/(2\pi R\gamma)} = \frac{\mathrm{d}s/\mathrm{d}\tau}{1 + (2\pi R^2\gamma\omega/c^2)(\mathrm{d}s/\mathrm{d}\tau)/(2\pi R\gamma)} \\ &= \frac{\mathrm{d}s/\mathrm{d}\tau}{1 + (\omega R/c^2)\,\mathrm{d}s/\mathrm{d}\tau} = \begin{cases} \frac{c}{1+\omega R/c} = \bar{c}_+ & \text{for counterclockwise motion} \\ \frac{-c}{1-\omega R/c} = -\bar{c}_- & \text{for clockwise motion} \end{cases}. \end{aligned} \tag{33}$$

So here we find the very satisfactory result that the speed of light is constant along the circumference in either direction (but not the same for both directions) and that its value is equal to the average speed (Eqs. 9,10) measured by Dorothy with a single clock (her own), a result that is obtainable without any synchronization.

With central synchronization, we do not have to explain, how it is possible that the average of a constant velocity can give a result different from the constant (which was due to the discontinuity of the time coordinate with local Einstein synchronization). So the description in terms of a centrally synchronized time coordinate is definitely simpler than that in terms of a locally Einstein synchronized one. It leads to a one-way speed of light that is different in

[26] That this gives the same synchrony as the resetting procedure based on the locally Einstein synchronized time, can be derived from its differential $\mathrm{d}\tilde{t} = \left(1 - \omega^2R^2/c^2\right)^{1/2}\left(\mathrm{d}t_I - ((\omega R^2/c^2)/(1-\omega^2R^2/c^2))\,\mathrm{d}\varphi\right)$, given in footnote 12. Here, $\mathrm{d}t_I$ is Inga's time differential. Because of $\Delta\tau_g = 2\pi R^2\gamma\omega/c^2 = 2\pi(\omega R^2/c^2)\left(1-\omega^2R^2/c^2\right)^{-1/2}$, we have $\mathrm{d}\tilde{t} = \left(1-\omega^2R^2/c^2\right)^{1/2}\mathrm{d}t_I - (\Delta\tau_g/2\pi)\,\mathrm{d}\varphi$ and $\mathrm{d}\tilde{\tau} = \mathrm{d}\tilde{t} + (\Delta\tau_g/2\pi)\,\mathrm{d}\varphi = \left(1-\omega^2R^2/c^2\right)^{1/2}\mathrm{d}t_I$, which means that $\mathrm{d}\tilde{\tau} = 0$ implies $\mathrm{d}t_I = 0$. That is, Inga's time and the time $\tilde{\tau}$ share the same simultaneity relation. They are not synchronous, because they run at different rates, but events that are simultaneous according to the time coordinate $\tilde{\tau}$ are simultaneous according to Inga's time. Clearly, the same effect is obtained by central synchronization, employing a signal by Inga that takes the same time to reach all the circumference observers.

the clockwise and counterclockwise directions. Since this is solely due to the chosen synchronization,[27] the two-way speed of light must still conform to Einstein's second postulate, and it does, as we have seen.

Does this mean that central synchronization is generally to be favored for the description of physics on a rotating disk? Or even, as Selleri might have put it (and Quattrini seems to believe) that "Nature prefers absolute synchronization".[28] Such a conclusion would be rash. As it turns out, the explanation of the Sagnac effect becomes simpler with central synchronization, but the description of many other physical phenomena becomes more difficult. This is the last topic that I would like to briefly touch upon.

With non-Einsteinian synchronization, the laws of Newtonian mechanics lose their familiar form. As has been shown by Ohanian [6], new kinds of pseudoforces then have to be introduced.

I give an example of a simple experiment performed on the circumference of the disk, the explanatory description of which is simple with (local) Einstein synchronization, but where central synchronization gives results that are difficult to interpret.

We take a metal tube, open at both ends, in the middle of which we have an anchoring peg to which two identical springs are attached, both of which are compressed to the same length by identical spherical balls (with a diameter slightly smaller than the inner diameter of the tube), held in place by two further pegs placed symmetrically with respect to the center of the tube. A trigger mechanism allows to retract these pegs outside of our tube through their boreholes in the tube walls. So the device we have constructed essentially is spring operated shotgun with the peculiarity that it shoots off two "bullets" at the same time but in opposite directions. Well, we have no desires to use it for warfare... We orient the tube tangentially to the disk circumference and shoot off our balls in the counterclockwise and clockwise directions (we may want to introduce guiding railings to keep them from leaving the disk, let these be friction free). What will the velocities of our two balls be? We can use the law of energy conservation to evaluate that. If the spring constants are k and the initial compression of the springs was Δx, then the potential energy of each spring before pulling the trigger was $k\Delta x^2/2$. Suppose that losses due to overshooting of the springs and oscillations are negligible and each spring transfers its complete potential energy to its respective ball. The balls have mass m each, so they will acquire the speed u given by $mu^2/2 = k\Delta x^2/2$, i.e., the potential energy of the springs has been turned into kinetic energy of the balls.

Now this description is valid in a frame with Einstein synchronization, as this particular form of the law of energy conservation can be derived within Newtonian mechanics in its standard form. But we know the outcome of the experiment in a centrally synchronized frame as well, because we know how to calculate velocities with central synchronization from velocities given with Einstein synchronization. In a centrally synchronized frame, our two balls will have different velocities. The one moving counterclockwise has a velocity $u_{cc} = u/(1+u\omega R/c^2)$, the other one $u_{cl} = u/(1 - u\omega R/c^2)$. Now here is a problem: the potential energies of the springs do not depend on time explicitly and are, therefore, independent of the synchronization; but the kinetic energies of the balls after they have been set in motion are different, because they have different speeds! From this one would conclude that either the law of energy conservation does not hold anymore or there must be forces acting on the balls in addition to those of the springs, one increasing the kinetic energy of the ball moving clockwise, the other decreasing that of the one moving counterclockwise.[29] Both options would mean that the description

[27] That is, it is not a consequence of acceleration.

[28] Or even imposes it upon us? (Forgetting for the moment that central synchronization and absolute synchronization are not quite the same thing.)

[29] These are pseudoforces, because their presence or absence depends on the choice of synchronization.

of the experiment using central synchronization is more complicated than that using (local) Einstein synchronization, with even the formulation of energy conservation not obvious in the former.

Ohanian has taken the fact that Newton's laws change their form with non-standard synchronization as evidence that Nature prefers, at least in inertial frames, Einstein synchronization. So there is another "proof" that only Einstein synchronization is the correct way of establishing simultaneity at a distance. Malament based his on mathematical and philosophical considerations, Ohanian on the idea that the familiar form of Newton's laws was deduced from the analysis of experiments.

Again, one does not have to subscribe to this. Apparently, when formulating mathematical descriptions of experiments, we automatically attribute properties to the time parameter that make it rather Einstein synchronized than synchronized in a more asymmetric way.[30] If we had a covariant description of Newton's laws in spacetime,[31] then we could of course automatically formulate the laws in arbitrary coordinates, which would include differently synchronized time coordinates. However, the fact that Einstein synchronization will give the simplest form of these laws, should not be mistaken as a compelling argument that this provides the only "physical simultaneity relation".

Neither should the increased simplicity of a description of light propagation around a rotating platform in terms of central synchronization be taken as an argument in favor of a "physical simultaneity" provided only by this or by some "absolute synchronization".

Rather, the correct view seems to be that simultaneity at a distance is not (entirely) physical but contains a conventional element. And of course, it is Einstein synchronization that makes the discussion of most physical situations and phenomena simplest in practice, with a few exceptions to be found in certain accelerating systems.

References

[1] K. Kassner, Ways to resolve Selleri's paradox, *Am. J. Phys.* **80**, 1061-1066 (2012).

[2] K. Kassner, Spatial geometry of the rotating disk and its non-rotating counterpart, *Am. J. Phys.* **80**, 772-781 (2012).

[3] A. Grünbaum, David Malament and the Conventionality of Simultaneity: A Reply. *Found. Phys.* **40**, 1285-1297 (2010).

[4] D. Malament, Causal Theories of Time and the Conventionality of Simultaneity, *Noûs* **11**, 293-300 (1977).

[5] G. Rizzi, M. L. Ruggiero, Space Geometry of Rotating Platforms: An Operational Approach, *Found. Phys.* **32**, 1525-1556 (2002).

[6] H. Ohanian, The role of dynamics in the synchronization problem, *Am. J. Phys.* **72**, 141-148 (2004).

[30] The so-called absolute synchronization, for example, would automatically introduce an asymmetry in the speed of light (and other velocities) between the directions parallel and antiparallel to the relative motion of the actual frame of reference and the absolute frame.

[31] The Lagrangian equations of the second kind are covariant with respect to arbitrary coordinate transformations not transforming the time, so they might be a starting point, from which to develop such a description.