Particle in a (homogeneous) gravitational field

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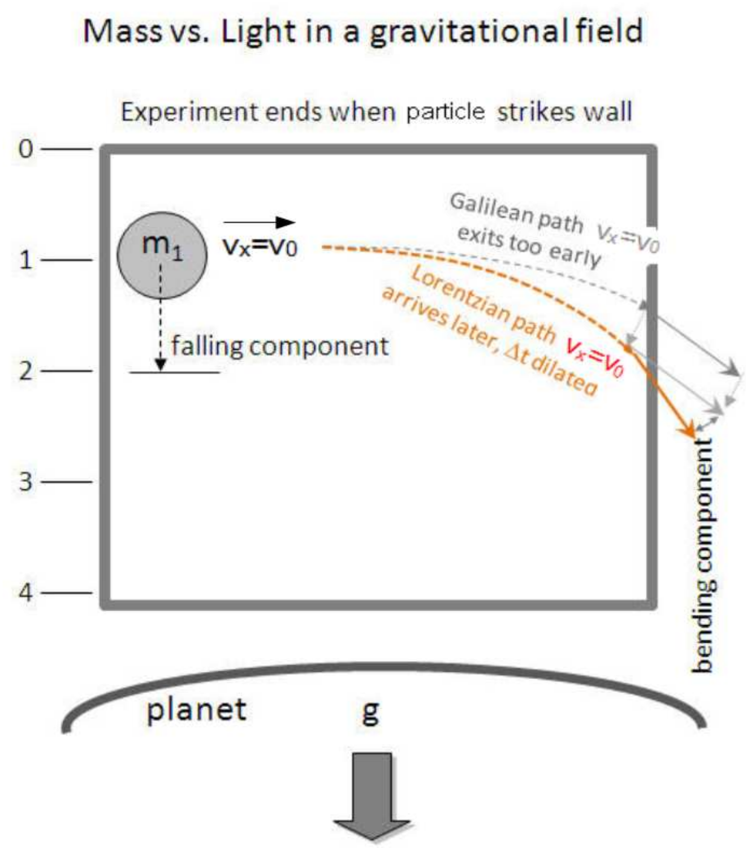

particle enters box horizontally at speed v_0; for photons, we simply have $v_0 = c$

Let the width of the box be L. We first do a Newtonian calculation of the trajectory, for reference purposes and as a zeroth-order approximation:

$$
\begin{aligned}
x &= v_0 t \,, \\
y &= -\frac{g}{2} t^2 = -\frac{g}{2v_0^2} x^2 \,.
\end{aligned}
$$

The total time to pass the box is, in this approximation, $\Delta t = \dfrac{L}{v_0}$, the total distance fallen by the particle is $\Delta y = -y(L) = \dfrac{g}{2v_0^2} L^2$. The ratio of the distances fallen by light ($v_0 = c$) and a massive particle with initial velocity $v_0 = v$ is simply $\frac{v^2}{c^2}$ in the Newtonian limit.

The exit angle is given by $\tan\phi = y'(L) = -\dfrac{gL}{v_0^2}$.

Now we wish to calculate the relativistic correction to this result. Since $g\Delta t \ll c$, unless the particle starts off very slowly, we anticipate the correction to be very small, so we expect it to be calculable perturbatively. In the following, we will assume $L \approx 1$ km and $v_0 \geq 10^{-6}c$. It will turn out that in this case the correction is small indeed.

We start with photons as for these the calculation is simplest. At each point of the trajectory, we must have:

$$\begin{aligned} v_x &= \cos\phi(x)\, c\,, \\ v_y &= \sin\phi(x)\, c\,. \end{aligned}$$

This looks as if it could be exact, if $\phi(x)$ were the exact angle between the tangent to the trajectory at x and the x axis. However, we do not know the exact trajectory. A good approximation will be obtained (as we can check afterwards) by substituting the Newtonian trajectory.

Hence,

$$\begin{aligned} \frac{v_y}{v_x} &= \tan\phi(x) = y'(x) \approx -\frac{gx}{c^2} \approx -\frac{gt}{c}\,, \\ c^2 &= v_x^2\left(1+\frac{v_y^2}{v_x^2}\right) \approx v_x^2\left(1+\frac{g^2t^2}{c^2}\right)\,, \\ v_x &= \frac{c}{\sqrt{1+\frac{g^2t^2}{c^2}}}\,, \qquad v_y = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}\,, \end{aligned}$$

Because of $gt \lesssim g\Delta t \ll c$, we may approximate the square root by its expansion:

$$v_x = \frac{c}{1+\frac{g^2t^2}{2c^2}}\,.$$

The total time Δt^* needed by the light to pass the box is then given by:

$$\begin{aligned} L &= \int_0^{\Delta t^*} v_x \mathrm{d}t = c\int_0^{\Delta t^*} \frac{1}{1+\frac{g^2t^2}{2c^2}}\,\mathrm{d}t \underset{u=\frac{gt}{\sqrt{2}c}}{=} \sqrt{2}\,\frac{c^2}{g}\int_0^{g\Delta t^*/\sqrt{2}c} \frac{\mathrm{d}u}{1+u^2} \\ &= \sqrt{2}\,\frac{c^2}{g}\arctan\frac{g\Delta t^*}{\sqrt{2}c}\,, \\ \frac{g\Delta t^*}{\sqrt{2}c} &= \tan\frac{Lg}{\sqrt{2}c^2} \approx \frac{Lg}{\sqrt{2}c^2}\left(1+\frac{1}{3}\left(\frac{Lg}{\sqrt{2}c^2}\right)^2\right)\,, \end{aligned}$$

where we have used $Lg/c^2 \ll 1$ to obtain the last simplification (this follows from our general assumption $g\Delta t \ll c$).

Hence, we obtain as lowest-order relativistic correction:

$$\boxed{\Delta t^* = \frac{L}{c}\left(1+\frac{1}{6}\frac{L^2g^2}{c^4}\right)}$$

and the distance fallen by the photon becomes in this approximation

$$\Delta y^* \approx -\frac{g}{2}\Delta t^{*2} = -\frac{g}{2}\frac{L^2}{c^2}\left(1+\frac{1}{6}\frac{L^2g^2}{c^4}\right)^2\,.$$

Alternatively, we may derive the result using conservation laws. The advantage of this approach is that it generalizes directly to massive particles.

First, we note that in a gravitational field with acceleration g, there is time dilation between the standard clocks of observers at different heights. For small height difference Δh, the ratio of the proper time intervals $\tau(h + \Delta h)$ and $\tau(h)$ is given by:

$$\frac{\tau(h+\Delta h)}{\tau(h)} = 1 + \frac{g\Delta h}{c^2}\,, \qquad\qquad [g = g(h)]\,,$$

i.e., at bigger heights people age faster. We take h to be the entry height of the particle, so Δh becomes equal to y.

If we take as time coordinate the proper time of a fixed observer, we have a homogeneous time, hence energy conservation holds. The energy of a photon, therefore its frequency, too, will not change for such an observer. But since time goes more slowly for an observer a distance $|\Delta h|$ down from the entry point, he will observe the photon at an increased frequency:[1]

$$\frac{\nu(h-|\Delta h|)}{\nu(h)} = \frac{1}{1-\frac{g|\Delta h|}{c^2}} \approx 1 + \frac{g\Delta h}{c^2}\,.$$

Next we note that space is homogeneous in the x direction, so the x component of momentum is conserved (the y component is not, because space is inhomogeneous in the y direction due to the gravitational field).

Then the x component of the photon momentum is given by

$$p_x = \frac{h\nu_0}{c} = \frac{h\nu(t)}{c}\cos\phi(t)\,.$$

But we know how ν changes with height:

$$\nu(t) = \nu_0\left(1 + \frac{g\,|y(t)|}{c^2}\right)\,,$$

which allows us to obtain $\cos\phi(t)$:

$$\cos\phi(t) = \frac{\nu_0}{\nu(t)} = \frac{1}{1+\frac{g|y(t)|}{c^2}} \approx \frac{1}{1+\frac{g^2t^2}{2c^2}}\,.$$

Since the horizontal component of the speed of light is

$$v_x = c\cos\phi(t) = \frac{c}{1+\frac{g^2t^2}{2c^2}}\,,$$

we obtain the same result as before.

[1]Note that for *both* observers, energy conservation holds, i.e., the observer at h considers the photon to always have the frequency $\nu(h)$, the one at $h - |\Delta h|$ always assigns the frequency $\nu(h - |\Delta h|)$ to it; they both agree that the other sees a different frequency, because her time passes faster or more slowly.

Now let us consider the case of a massive particle. Here, conservation of the x component of momentum reads:

$$m\gamma(v_0)v_0 = m\gamma(v)v_x \,, \qquad\qquad \gamma(v) = \frac{1}{1 - v^2/c^2} \,,$$

where m is the *invariant* mass (the "inertial mass" is $m\gamma(v)$).

Energy conservation means that $E = m\gamma(v)c^2$ is constant for an observer at fixed height. An observer at a lower height will see an increased energy due to time dilation

$$\frac{E(h - |\Delta h|)}{E(h)} = \frac{1}{1 - \frac{g|\Delta h|}{c^2}} \approx 1 + \frac{g\Delta h}{c^2} \,.$$

Energy and frequency behave absolutely the same under time dilation, which means that the inertial mass, which is just E/c^2 is constant for an observer at fixed height in the case of a freely falling particle.[2]

This means we can directly express $\gamma(v)$ by the height difference

$$\frac{E(y)}{E(0)} = \frac{m\gamma(v)c^2}{m\gamma(v_0)c^2} = \frac{\gamma(v)}{\gamma(v_0)} = 1 + \frac{g\,|\Delta y|}{c^2} \,.$$

Momentum conservation then simplifies to

$$v_0 = \left(1 + \frac{g\,|\Delta y|}{c^2}\right) v_x \,,$$
$$v_x = \frac{v_0}{1 + \frac{g|\Delta y|}{c^2}} \approx \frac{v_0}{1 + \frac{g^2t^2}{2c^2}} \,.$$

The total time for the particle to cover the length L is then obtained via

$$L = \int_0^{\Delta t^*} v_x \mathrm{d}t = v_0 \int_0^{\Delta t^*} \frac{1}{1 + \frac{g^2t^2}{2c^2}} \,\mathrm{d}t = \sqrt{2}\,\frac{cv_0}{g} \arctan \frac{g\Delta t^*}{\sqrt{2}c} \,,$$
$$\frac{g\Delta t^*}{\sqrt{2}c} = \tan\left(\frac{Lg}{\sqrt{2}cv_0}\right) \approx \frac{Lg}{\sqrt{2}cv_0}\left(1 + \frac{1}{3}\left(\frac{Lg}{\sqrt{2}cv_0}\right)^2\right) \,.$$

Our final result for the time of flight is then

$$\boxed{\Delta t^* = \frac{L}{v_0}\left(1 + \frac{1}{6}\frac{L^2g^2}{c^2v_0^2}\right)}$$

and the distance fallen becomes

$$\Delta y^* \approx -\frac{g}{2}\Delta t^{*2} = -\frac{g}{2}\frac{L^2}{v_0^2}\left(1 + \frac{1}{6}\frac{L^2g^2}{c^2v_0^2}\right)^2 \,.$$

[2] If the particle is prevented from freely falling, e.g. by being mounted on a track that is lowered in the gravitational field, its kinetic energy cannot increase as it would in free fall. So its total energy and hence inertial mass decreases from the point of view of the fixed-height observer, who can extract the energy difference on lowering the particle slowly.

Let us call $\Delta y^*_{\rm mp}$ the distance fallen by the massive particle and $\Delta y^*_{\rm ph}$ the distance fallen by the photon. We then obtain for the ratio of these distances:

$$\frac{\Delta y^*_{\rm mp}}{\Delta y^*_{\rm ph}} = \frac{c^2}{v_0^2}\left(\frac{1+\frac{1}{6}\frac{L^2g^2}{c^2v_0^2}}{1+\frac{1}{6}\frac{L^2g^2}{c^4}}\right)^2 .$$

The first factor c^2/v_0^2 corresponds to the Newtonian result and the second factor is the lowest-order special relativistic correction. Since this factor is not equal to 1, there is indeed such a correction, let us call it Shuler's correction. We have to check now whether our approximations were o.k., i.e. whether the corrction is small. To this end, we plug in reasonable numbers.

We consider a box of size $L = 1$ km and we take $g = 274\ \frac{\rm m}{\rm s^2}$, which is the acceleration at the surface of the sun (taken from Wikipedia). For v_0 we take $10^{-6}c$.

This produces

$$\frac{Lg}{c^2} = 3\times 10^{-12}\,, \qquad \left[\Rightarrow \qquad v_{{\rm ph}\,x}(\Delta t^*) \approx \frac{c}{1+\frac{g^2L^2}{2c^4}} = c\left(1-\frac{9}{2}\times 10^{-24}\right)\right]$$

$$\frac{Lg}{cv_0} = 3\times 10^{-6}\,, \qquad \left[\Rightarrow \qquad v_{{\rm mp}\,x}(\Delta t^*) \approx \frac{c}{1+\frac{g^2L^2}{2c^2v_0^2}} = v_0\left(1-\frac{9}{2}\times 10^{-12}\right)\right]$$

pretty small numbers indeed, leading to

$$\frac{\Delta y^*_{\rm mp}}{\Delta y^*_{\rm ph}}\frac{v_0^2}{c^2} = \left(\frac{1+\frac{1}{6}\times 9\times 10^{-12}}{1+\frac{1}{6}\times 9\times 10^{-24}}\right)^2 \approx 1+3\times 10^{-12}\,.$$

Hence, our approximations were consistent, the result is quantitative.

So the good news is: There is indeed a Shuler's correction to the equivalence principle part of the deflection angle. The bad news is that Einstein was completely right in neglecting it. As one goes farther away from the sun, the correction will be of comparable size in each box first but then become smaller with increasing distance. Therefore, the total angle of deflection will be corrected by a similar or smaller factor, i.e., by an order of magnitude of 10^{-10} percent, which is certainly below what is measurable at this moment.

It should also be observed that most of the correction comes from the relativistic slowing down of the massive particle, not from that of light ($Lg/cv_0 \gg Lg/c^2$), so the main effect is different from what was anticipated by Robert Shuler.

Another good news is of course that the result, which does not double the standard prediction but only changes it by a tiny tiny bit, saves us from the embarrassment of having to explain why the total angle of deflection including the general relativistic part (that is not obtainable from the equivalence principle), is not 3/2 of what is observed right now.