Is the electrostatic force between a point charge and a neutral metallic object always attractive?

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We give an example of a geometry in which the electrostatic force between a point charge and a neutral metallic object is repulsive. The example consists of a point charge centered above a thin metallic hemisphere, positioned concave up. We show that this geometry has a repulsive regime using both a simple analytical argument and an exact calculation for an analogous two-dimensional geometry. Analogues of this geometry-induced repulsion appear in many other contexts, including Casimir systems.
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I. INTRODUCTION

A classic problem in electrostatics is the calculation of the force between a point charge and a perfectly conducting, neutral metallic sphere [see Fig. 1(a)]. The force on the point charge can be calculated by summing the forces exerted by two image charges—one located at the center of the sphere, carrying like charge, and one closer to the surface, carrying the opposite charge.

A simple consequence of this analysis is that the force is always attractive, because the oppositely charged image charge is always closer to the point charge than its partner. This attraction makes sense intuitively, because we expect that a positively charged point charge induces negative charges on the part of the sphere that is closest to it and positive charges on the part that is further away. It is natural to wonder if this phenomenon is more general and if the force is attractive for any geometry, not just a sphere.

This question is the main subject of this paper. We can think of this question as an attempt to strengthen Earnshaw's theorem. Recall that Earnshaw's theorem and its generalizations tell us that a point charge can never be trapped in a stable equilibrium via electrostatic interactions with a metallic object.[1] Here, we ask whether we can go further for the case where the metallic object is neutral and show that the force is always attractive.

To make the question precise, we need to define what we mean by an "attractive" force. To this end, it is useful to make the additional assumption that the charge and metallic object lie on opposite sides of a plane, say, the $z=0$ plane, with the charge in the upper half space $(z>0)$ and the metal object in the lower half space $(z<0)$ [see Fig. 1(b)]. By an attractive force, we mean a force $\mathbf{F}$ on the charge with $F_z<0$.

Given this definition, it is not difficult to show that the force is attractive in several cases. We first consider a point charge that is very close to the surface of the metal object. In this case, the problem reduces to the standard system of a charged particle and an infinite metal plate, which clearly has an attractive force. Another case is when the point charge is very far from the metallic object—say at position $(0,0,z)$, where z is large. To see that the force is attractive in this case, recall Thomson's theorem: the induced charges in a metallic object always arrange themselves to minimize the total electrostatic energy of the system.[2] A corollary of this theorem is that the electrostatic energy of a system composed of a metallic object and a charge is always lower than the energy of the charge in vacuum. We let $U(z)$ denote the electrostatic energy when the charge is at position $(0,0,z)$ and conclude that $U(z)\le U(\infty)$ so that $F_z=-dU/dz$ must be negative (attractive) for large z. A third case involves a related problem where the metal object is replaced by a dielectric material with a dielectric constant $\varepsilon=1+\delta$ with $0<\delta\ll 1$. One way to see that the force is attractive in this geometry is to note that, to lowest order in δ, the electrostatic interaction between the charge and the object can be decomposed into a sum of independent interactions with infinitesimal patches of dielectric material. We can then check that each patch gives rise to an attractive interaction, so that the total interaction is necessarily attractive. As a final example, we can show that the force is attractive if the metallic object is grounded rather than neutral. In this case, a positively charged point charge induces only negative charges on the metallic object, leading to an attractive force.

Given all of these examples, we might conclude that the force is always attractive. Surprisingly, this conclusion does not hold. In this paper, we give a simple example of a geometry in which a neutral metallic object repels a point charge. We establish repulsion using both a simple analytical argument and an exact calculation for an analogous two-dimensional (2D) geometry. In accordance with Earnshaw's theorem and its generalizations,[1] our geometry does not yield any stable equilibria. However, the fact that we can have repulsion at all is surprising, and we show that analogues of this unusual geometric effect exist in several other contexts, including Casimir systems.

II. EXAMPLE OF A REPULSIVE GEOMETRY

The geometry that gives a repulsive force consists of an thin metallic hemisphere of radius R, centered at the origin, and positioned in the lower half space $z<0$, together with a point charge at position $(0,0,z)$ on the positive z axis (see Fig. 2). The system is cylindrically symmetric about the z axis, so the force that the hemisphere exerts on the charge necessarily points in the z direction. When the charge is far

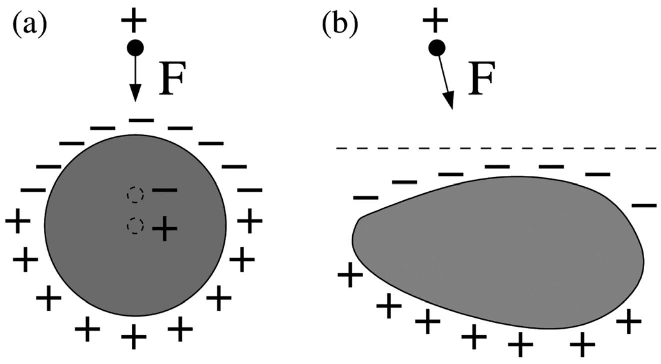

Fig. 1. (a) Electrostatics of a point charge interacting with a neutral metallic sphere. By using image charges (dotted circles), it is easy to see that the force **F** on the point charge is attractive. (b) We ask whether the force is attractive for any shape of metallic object. More precisely, if the point charge and the object are on opposite sides of the $z = 0$ plane (dotted line), with the charge in $\{z > 0\}$ and the object in $\{z < 0\}$, is F_z always negative?

from the hemisphere, that is, $z \gg R$, the force is necessarily attractive by the general argument given previously. We now show that the force changes sign and becomes repulsive when the charge approaches the $z = 0$ plane.

We can establish the existence of a repulsive regime without any calculation if we assume an idealized geometry where the hemisphere is infinitesimally thin. The idea is to consider the case where the point charge is at the origin, $z = 0$. When the point charge is at this special point, the Coulomb electric field lines of the point charge are all perpendicular to the hemisphere [see Fig. 3(a)]. In other words, the hemisphere is an equipotential surface of the Coulomb potential, $\phi(\mathbf{r}) = q/r$, which means that the Coulomb potential solves the relevant boundary value problem. By the uniqueness theorem in electrostatics, there can be only one solution to this boundary value problem, so the solution must be exactly the Coulomb potential and Coulomb electric field. (As a historical aside, we note that the idea of placing thin conductors on equipotential surfaces created by a known charge distribution is an approach that goes back to Maxwell.)[3–5]

Using this observation, we now argue that when the charge is at $z = 0$, the electrostatic energy U of the system is the same as when the charge is at $z = \infty$. We deduce this equality by comparing the electric field distributions in these two cases. When the charge is at $z = 0$, the field lines are given by the Coulomb electric field $\mathbf{E} = q\hat{r}/r^2$, as argued previously. When the charge is at infinity, the metallic hemisphere has no effect, and thus the electric field distribution is

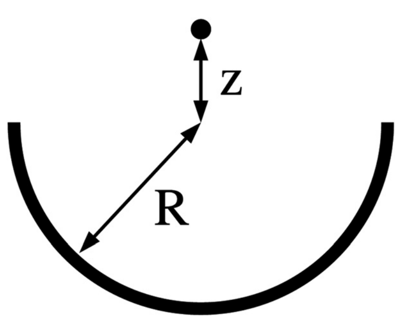

Fig. 2. Example of a geometry in which a neutral metallic object repels a point charge: a point charge centered above a thin metallic hemisphere (side view).

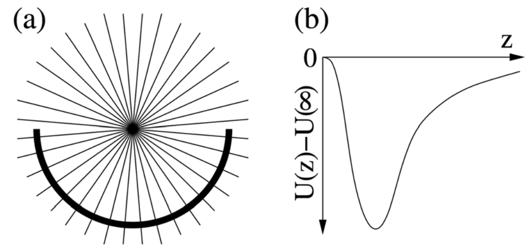

Fig. 3. Argument that an infinitesimally thin metallic hemisphere repels a point charge. (a) At $z = 0$, the vacuum electric field lines of the point charge are already perpendicular to the hemisphere (side view), and thus the electric field is unaffected by the presence of the hemisphere. (b) Schematic charge-hemisphere interaction energy $U(z) - U(\infty)$. Because it is zero at $z = 0$ and at $z \to \infty$ and attractive for $z \gg R$, there must be repulsion for small positive z.

also given by the Coulomb electric field. In particular, we see that the two field distributions are identical. We then conclude that, because the electrostatic energy U of the system is given by

$$U = \frac{1}{8\pi}\int \mathbf{E}^2 d^3\mathbf{x}, \tag{1}$$

which depends only on the electric field distribution $\mathbf{E}$, the electrostatic energy of the system must be identical when the charge is at $z = 0$ and at $z = \infty$.[6] In other words, if we interpret U as a function of the charge's position z, then $U(0) = U(\infty)$.

The existence of a repulsive regime follows immediately from the equality $U(0) = U(\infty)$: $U(z)$ must vary non-monotonically between $z = 0$ and $z = \infty$ and in particular must be decreasing, that is, repulsive, at some intermediate points [see Fig. 3(b)]. We can even go a bit further and argue that there must be a repulsive regime when z is small and positive. Recall the inequality $U(z) \le U(\infty)$ derived in Sec. I. Because $U(0) = U(\infty)$, we have $U(z) \le U(0)$, which implies that $F_z = -dU/dz$ is positive (repulsive) for small positive z.

Because the interaction is attractive for large z and repulsive for small z, the simplest consistent scenario is that the electrostatic energy $U(z) - U(\infty)$ is zero at $z = 0$, decreases to negative values for small $z > 0$, and then increases to zero for large z, as depicted in Fig. 3(b). We confirm this scenario in Sec. IV with an exact solution of an analogous 2D system. Note that the point of minimum U is an equilibrium position, stable under perturbations in the z direction. By Earnshaw's theorem and its generalizations,[1] this equilibrium point must be unstable to lateral (xy) perturbations.

So far, we have focused on an idealized geometry where the hemisphere is infinitesimally thin. Now suppose that the hemisphere has a finite thickness t. In this case, the hemisphere no longer lies on an equipotential surface for the Coulomb potential, and hence the Coulomb electric field no longer gives an exact solution to the boundary value problem. Therefore, the previous argument cannot be applied directly. However, as long as t/R is small, the electrostatic energy curve [Fig. 3(b)] can shift only by a small amount from the $t = 0$ case, so that the repulsive regime must continue to persist. To make this argument more quantitative, note that the main effect of the finite thickness is to expel the electric field from the volume of the hemisphere, $V = 2\pi R^2 t$. As a result, we have to lowest order in t

$$U(0) - U(\infty) = -\frac{V}{8\pi}\mathbf{E}^2 = -\frac{q^2 t}{4R^2}, \tag{2}$$

where q is the charge carried by the point charge. We compare Eq. (2) with the minimum value of U, which is of order $U_{\min} - U(\infty) \sim -q^2/R$ by dimensional analysis and see that $U_{\min} < U(0)$ for small t/R. Hence, the repulsive regime must persist in the presence of small, finite thickness. In contrast, when t/R becomes sufficiently large, the repulsive regime disappears completely, as we explain in Sec. III. Numerical calculations for an analogous 2D geometry confirm this picture (see Fig. 5).

III. GEOMETRIC ORIGIN OF THE REPULSION

The general argument we have described shows that there must be a repulsive regime, but it does not tell us what causes the repulsion. To address this question, it is useful to consider the induced charges on the hemisphere when the charge is at $(0, 0, z)$ on the positive z axis. In general, there will be charges on both sides of the hemisphere, but in the limit where the hemisphere is very thin, we can make the approximation of combining the charges on the two sides into a single surface charge density σ. If we assume that the point charge is positive, we expect this total charge density to be of the form shown in Fig.4(a), with σ positive in the center of the hemisphere and negative near the boundary. We would like to understand the force that these induced charges exert on the point charge. Clearly, the negative charges are closer to the point charge than the positive charges, so they exert a stronger force on it. Naively, we might expect this difference in distances to lead to a net attractive force. However, the key point is that the angle between the force direction and the z axis is smaller for the positive charges than the negative charges, and thus even though they are further away, they can potentially exert a greater force in the z direction, depending on the position of the point charge.

More precisely, the z component of the force that a charge on the hemisphere exerts on the point charge is proportional to $\cos\theta/r^2$, where r is the distance to the charge and θ is the angle with respect to the z axis [see Fig. 4(a)]. The positive charges have a larger r, but also a larger value of $\cos\theta$ than the negative charges. The competition between these two geometrical effects determines the sign of the force. If the point charge is very close to the origin, $z \ll R$, then the trigonometric factor $\cos\theta$ dominates; r is almost the same for the positive and negative charges ($r \approx R$), but $\cos\theta$ is larger for the positive charges. The result is a repulsive force. In contrast, if the point charge is very far away, $z \gg R$, then the $1/r^2$ factor is larger for the negative charges by a factor of size $1 + O(R/z)$, while $\cos\theta$ is almost the same for the positive and negative charges, that is, $\cos\theta = 1 + O(R^2/z^2)$. The result is an attractive force.

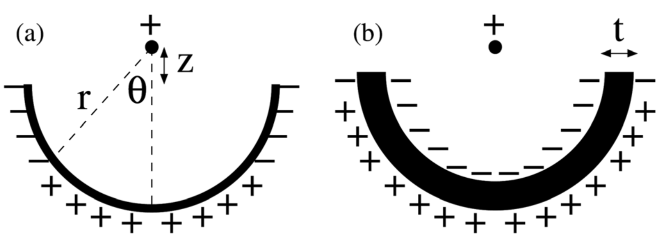

Fig. 4. (a) Schematic charge density σ induced by a point charge on an infinitesimally thin hemisphere (side view). The force that an induced charge on the hemisphere exerts on the point charge in the z direction is proportional to $\cos\theta/r^2$. The positive charges have a larger r then the negative charges, but also a larger $\cos\theta$. The latter effect dominates and leads to a repulsive force for small, positive z. (b) Induced charge density on a hemisphere with finite thickness t. The displacement between the two surface densities makes an attractive contribution to the force and destroys the repulsive regime if the thickness t is large.

This argument also explains why the repulsive regime disappears when the thickness t becomes comparable to R. Once t/R is appreciable, we can no longer make the approximation of combining the charges on the two sides of the hemisphere into a single charge density. Instead, we need to treat the two surface charge densities separately. Intuitively, we expect that the charges on the inner surface are mostly negative, while the charges on the outer surface are mostly positive, as depicted in Fig. 4(b). The finite distance between the two surfaces makes an attractive contribution to the total force, because the negative charges are closer to the point charge than the positive charges. This effect can overwhelm the $\cos\theta$ factor when t/R is sufficiently large, destroying the repulsive regime completely.

IV. EXACT SOLUTION IN TWO DIMENSIONS

To gain additional insight, we next calculate the force exactly in an analogous 2D geometry. The geometry consists of a metal semicircle of radius R, which we denote by S_R, together with a point charge. In analogy with the three-dimensional (3D) case, we take the semicircle to be centered at the origin and positioned in the lower half plane. More precisely, in terms of the Cartesian coordinates (x_1, x_2), we take S_R to be the set of all points with $|x_1|^2 + |x_2|^2 = R^2$ and $x_2 \le 0$. We take the point charge to be at position $\mathbf{y} = (0, z)$ with $z > 0$.

We now calculate the 2D electrostatic interaction between the point charge and the metal semicircle assuming that the point charge carries charge q. Our starting point is the boundary value problem defined by

$$\nabla^2 \phi_{\mathbf{y}}(\mathbf{x}) = -2\pi q \delta(\mathbf{x} - \mathbf{y}), \tag{3}$$

with the boundary conditions

$$\phi_{\mathbf{y}}(\mathbf{x}) = \text{constant} \qquad (\mathbf{x} \in \partial S_R), \tag{4}$$

$$\phi_{\mathbf{y}}(\mathbf{x}) + q\log|\mathbf{x} - \mathbf{y}| = 0 \quad \text{for} \quad \mathbf{x} \to \infty, \tag{5}$$

$$\int_{\partial S_R} \mathbf{n} \cdot \nabla\phi_{\mathbf{y}}(\mathbf{x}) d\mathbf{x} = 0. \tag{6}$$

Equation (4) imposes the boundary condition that the semicircle is an equipotential surface, and Eq. (6) imposes the condition that the semicircle is electrically neutral. Equation (5) requires the potential (8) to vanish at infinity. The force that the metallic object exerts on the charge is given by

$$\mathbf{F}(\mathbf{y}) = -q\nabla\tilde{\phi}_{\mathbf{y}}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}}, \tag{7}$$

where

$$\tilde{\phi}_{\mathbf{y}}(\mathbf{x}) = \phi_{\mathbf{y}}(\mathbf{x}) + q\log|\mathbf{x} - \mathbf{y}|, \tag{8}$$

is the potential created by the induced charges on the metal object. The electrostatic energy of the system, $U(\mathbf{y})$, is given by

$$U(\mathbf{y}) - U(\infty) = -\int_{\infty}^{\mathbf{y}} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} \tag{9}$$

$$= \frac{q}{2} \tilde{\phi}_{\mathbf{x}}(\mathbf{x})\big|_{\infty}^{\mathbf{y}} = \frac{q}{2} \tilde{\phi}_{\mathbf{y}}(\mathbf{y}). \tag{10}$$

The second equality follows from the fact that $\tilde{\phi}_{\mathbf{x}}(\mathbf{y}) = \tilde{\phi}_{\mathbf{y}}(\mathbf{x})$ so that $(q/2)\nabla_x \tilde{\phi}_{\mathbf{x}}(\mathbf{x}) = -\mathbf{F}(\mathbf{x})$.

Our strategy will be to solve the boundary value problem (3) using a conformal mapping, obtain $\tilde{\phi}_{\mathbf{y}}$, and then calculate the energy (10). To this end, we view our 2D system as the complex plane $\mathbf{C}$, and use complex coordinates $u = x_1 + ix_2$, $v = y_1 + iy_2$ in place of $\mathbf{x}$, $\mathbf{y}$. We can check that the analytic function

$$h(u) = \frac{iR + u + i\sqrt{R^2 - u^2}}{2}, \tag{11}$$

defines a conformal map from the region outside the semicircle S_R to the region outside the disk D of radius $R/\sqrt{2}$ centered at the origin. In other words, h maps the set of all points $u = x_1 + ix_2$ with either $x_1 > 0$ or $|u| \neq R$ onto the set of points with $|u| > R/\sqrt{2}$. Following the conformal mapping approach to 2D boundary value problems,[7] the function h allows us to map the boundary value problem corresponding to a charge interacting with a metallic disk D onto the problem of a charge interacting with a metallic semicircle S_R.

The boundary value problem for a metallic disk can be easily solved using image charges. The potential for this geometry is given by

$$\phi_v^D(u) = -q \log|u - v| + q \log\left|u - \frac{R^2}{2\bar{v}}\right| - q \log|u|. \tag{12}$$

It follows that the potential for the semicircle geometry is

$$\phi_v(u) = \phi^D_{h(v)}(h(u)) \tag{13}$$

$$= -q\log|h(u) - h(v)| + q\log\left|h(u) - \frac{R^2}{2h(\bar{v})}\right| - q\log|h(u)|, \tag{14}$$

so that

$$\tilde{\phi}_v(v) = -q\log\left|\frac{dh}{dv}\right| + q\log\left(1 - \frac{R^2}{2|h(v)|^2}\right). \tag{15}$$

We substitute the expression for h in Eq. (11) and derive

$$\tilde{\phi}_v(v) = q\log\left(\frac{2 - \dfrac{4R^2}{|iR + v + i\sqrt{R^2 - v^2}|^2}}{\left|1 - i\dfrac{v}{\sqrt{R^2 - v^2}}\right|}\right). \tag{16}$$

If we specialize to the case where v is on the positive imaginary axis, $v = iz$, so that $\mathbf{y} = (0, z)$, and use the convention that $U(\infty) = 0$, we obtain the electrostatic energy

$$U(z) = \frac{q^2}{2}\log\left(\frac{2 - \dfrac{4R^2}{(R + z + \sqrt{R^2 + z^2})^2}}{1 + \dfrac{z}{\sqrt{R^2 + z^2}}}\right). \tag{17}$$

A plot of $U(z)$ is shown in Fig. 5, together with numerical results for finite thickness geometries included for comparison. We can see from Fig. 5 (or from a little algebra) that the force $F_z = -dU/dz$ is repulsive for $0 < z < R$ and attractive for $z > R$, in agreement with our previous arguments. Using Eq. (16), we can check that the equilibrium at $z = R$ is unstable to perturbations away from the symmetry axis, as required by Earnshaw's theorem and its generalizations.[1]

We can also see from Fig. 5 that the force changes sign again at $z = 0$ (for zero thickness). The existence of an additional sign change is not surprising because as $z \to -R$, the force must approach the attractive interaction between a charge and an infinite metal plate. As for why the sign change occurs exactly at $z = 0$, there are several ways to understand this apparent coincidence. One way is to note that the general inequality $U(z) \le U(\infty)$, together with the fact that $U(0) = U(\infty)$, implies that U reaches a maximum at $z = 0$. Another way is to note that the induced charges on the semicircle arrange themselves oppositely when $z < 0$ as compared with $z > 0$ [see Fig. 4(a)], because when $z < 0$, the center of the semicircle is closer to the point charge than the boundary, and the situation is reversed for $z > 0$. This sign change in the induced charge arrangement implies a corresponding sign change in the force. Alternatively, we note that when $z = 0$, there are no induced charges (or more accurately, the induced charges on the two sides of the semicircle cancel exactly), so the force necessarily vanishes.

The electrostatic energy for the 3D charge-hemisphere geometry, $U_{3D}(z)$can also be obtained analytically, although the calculation[8] is more complicated. For completeness, we include a plot of the result $U_{3D}(z)$ in Fig. 6. Note the similarity to the 2D energy $U(z)$, though in three dimensions, the force changes sign at $z \approx 0.63R$ rather than at $z = R$.

The vanishing of F_z at $z = R$ in the 2D case can be established without any calculation, because it follows from a simple geometrical argument similar to the one in Sec. III. Consider the z component of the force that an induced charge on the semicircle exerts on the point charge. In analogy with Fig. 4(a), this quantity is proportional to $\cos\theta/r$, where r is

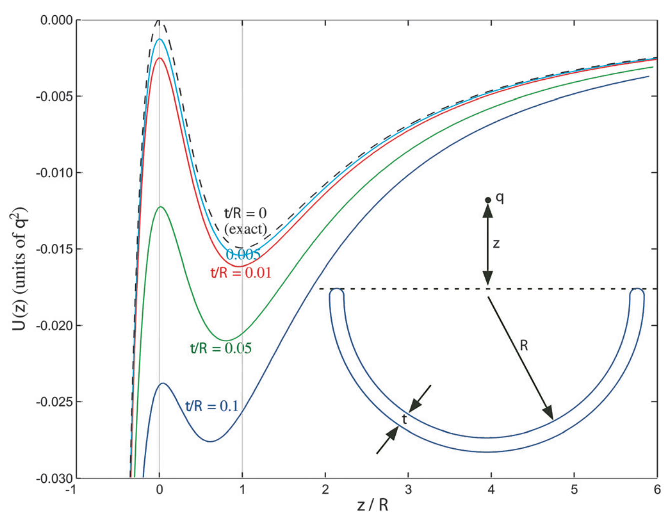

Fig. 5. (Color online) The dashed curve is the exact (analytic) 2D electrostatic interaction energy $U(z)$ for the charge-semicircle geometry in the limit of zero thickness. The solid curves are numerical results for finite-thickness ($t \neq 0$) geometries for $t/R = 0.1$, 0.05, 0.01, and 0.005 from bottom to top (inset: $t/R = 0.1$). The numerical results used a boundary-element surface-integral-equation method,[13] Nyström-discretized with 12000 points for better than 1% accuracy.

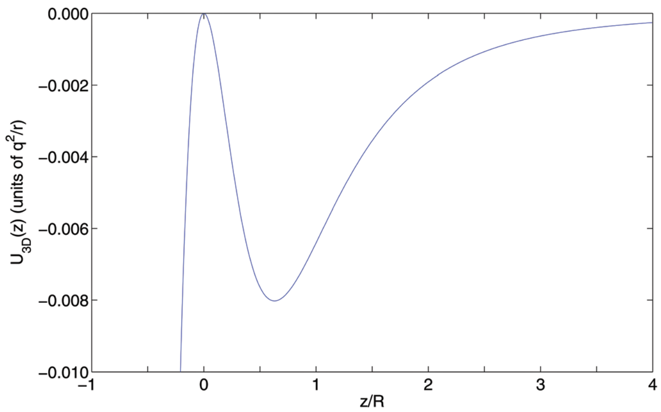

Fig. 6. (Color online) Exact (analytic) electrostatic interaction energy $U_{3\mathrm{D}}(z)$ for charge-hemisphere geometry.

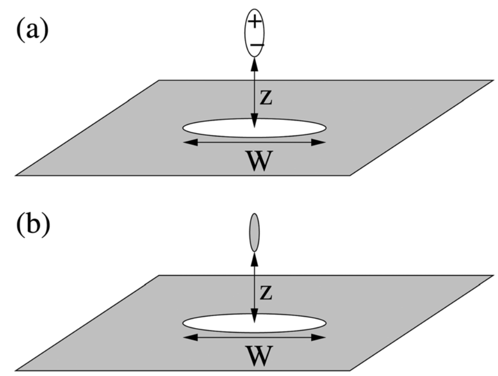

Fig. 7. (a) Example of a geometry in which a neutral metallic object repels an electric dipole: a dipole centered above a thin metallic plate with a hole. (b) Example of a geometry achieving Casimir repulsion: an elongated metal particle centered above a thin metal plate with a hole.

the distance to the point charge and θ is the angle with respect to the z axis. For most locations of the point charge, this geometrical factor varies from place to place on the semicircle, so that the force that an induced charge exerts on the point charge depends on where it is located. However, when the point charge is exactly at $(0, R)$, a little geometry shows that $\cos\theta/r \equiv 1/(2R)$ for every point on the semicircle. This geometrical identity means that all the induced charges exert the same force in the z direction. Because the object is neutral, the contributions from the positive and negative induced charges cancel exactly, and we conclude that $F_z = 0$. We emphasize that this argument is specific to the 2D case. In the 3D charge-hemisphere geometry, the force changes sign at $z \approx 0.63R$ rather than $z = R$. We do not know of an analogous geometrical derivation of the crossover point in three dimensions.

V. RELATED PHENOMENA

A. A metallic object that repels an electric dipole

In this section, we give an example of another unusual electrostatic geometry: a metallic object that repels an electric dipole. In most cases, the interaction between a dipole and a metallic object is attractive, and again we need a special geometry to get a repulsive force.

The counterexample geometry consists of a metallic plate with a circular hole of diameter W, located in the $z = 0$ plane and centered at the origin, together with a z-directed dipole at position $(0, 0, z)$[see Fig. 7(a)]. To see that this system has a repulsive regime, we use the same argument as before. We consider the special case where the dipole is located at the origin, $z = 0$. When the dipole is at this special point, the vacuum dipole field lines are all perpendicular to the metal plate, which means that the vacuum electric field solves the relevant boundary value problem. Because the electric field for $z = 0$ is identical to the field in vacuum ($z = \infty$), we conclude that the energy U is also identical and $U(0) = U(\infty)$. As before, this equality implies that the energy is non-monotonic and hence must be repulsive at some intermediate points. Note that the key property of this geometry is that the metal plate is an equipotential surface for the dipole at $z = 0$, just as the hemisphere was an equipotential surface for a point charge at $z = 0$.

Again we expect the force to be attractive for large z and repulsive for small z, so that the energy $U(z) - U(\infty)$ is of the form shown in Fig. 3(b). We can confirm this picture by exactly solving a 2D analogue of this geometry (the 3D case can also be solved exactly, though the calculation[9] is more involved). The 2D analogue consists of a metal line with a gap of width W together with an electric dipole. In terms of Cartesian coordinates, (x_1, x_2), we take the metal line to be the set of all points with $x_2 = 0$ and $x_1 \geq W/2$, and we take the electric dipole to be located at $(0, z)$, oriented in the z-direction. A conformal mapping approach similar to the one in Sec. IV gives

$$U(z) = -p_z^2 \frac{2z^2}{(W^2 + 4z^2)^2}, \tag{18}$$

where we use the convention $U(\infty) = 0$. By taking the derivative with respect to z, we find that the force is attractive for $z > W/2$ and repulsive for $0 < z < W/2$. As in the point charge case, we can show that the equilibrium at $z = W/2$ is unstable to perturbations away from the symmetry axis, as required by Earnshaw's theorem and its generalizations.[1]

B. A geometry with a repulsive Casimir force

The Casimir force arises from quantum fluctuations in the electric and magnetic polarizations of matter.[10] It can be regarded as a generalization of the van der Waals force to include retardation effects. Most famously, it gives rise to an attractive interaction between parallel neutral metallic plates in vacuum.

A longstanding question is whether the Casimir force between metallic objects in vacuum is always attractive. If we use the dipole-metallic object system discussed in Sec. V B, we can answer this question in the negative and construct a simple repulsive geometry for the Casimir force. In the following, we will describe the geometry and briefly explain why it is repulsive and how it is related to the dipole system. A more detailed discussion can be found in Ref. 11.

The repulsive Casimir geometry consists of a metallic plate with a circular hole of diameter W, located in the $z = 0$ plane and centered at the origin, together with an elongated metallic particle at $(0, 0, z)$, oriented with the long axis in the z direction [see Fig. 7(b)]. Our claim is that this geometry has a repulsive regime in the limit that the particle is infinitesimally small and highly elongated (the limit of an infinitesimal metallic needle.)

To understand this claim, note that the Casimir interaction can be thought of as a electromagnetic interaction between zero-point quantum mechanical charge fluctuations on the particle and the associated induced charges on the plate. Because the particle is highly elongated and infinitesimally small, the only charge fluctuations are z-directed dipole fluctuations. Hence, the problem reduces to understanding the classical electromagnetic interaction between these z-directed dipole fluctuations and the plate with a hole.

The argument now proceeds as in the electrostatic case. We consider the special case where the particle is located at the origin, $z = 0$. When the particle is at this special point, its dipole fluctuations do not couple to the plate at all, because the vacuum dipole field lines are already perpendicular to the plate. This decoupling holds for not only zero frequency dipole fluctuations (as discussed in Sec. V A), but also for finite frequency fluctuations. The decoupling between the dipole fluctuations and the plate is guaranteed by symmetry because the metal plate is symmetrical with respect to the $z = 0$ mirror plane, while the dipole fluctuations are antisymmetric. Because the particle and plate do not couple, it follows that the Casimir energy at $z = 0$ is the same as at infinite separation, $U(0) = U(\infty)$, so that the energy must vary non-monotonically, and hence must be repulsive at some intermediate points.

For $z \gg W$, the hole in the plate can be neglected, and we have the usual attractive interaction. Therefore, we expect the interaction energy to be of the form shown in Fig. 3(b), with a repulsive regime for small z, an attractive regime for large z, and a sign change for at some $z \sim W$. This expectation is confirmed by numerical calculations.[11]

As in the electrostatic examples, the point of minimum U is an unstable equilibrium as the particle is unstable to perturbations away from the symmetry axis. Thus, this geometry does not support stable Casimir levitation. This instability is consistent with the instability theorem of Ref. 12—an analogue of Earnshaw's theorem for the Casimir force.

C. Current flow analogues

In this section, we discuss analogues of these geometrical effects involving current flow in a resistive sheet. We show that current flows can behave in counterintuitive ways in certain geometries. Our starting point is a perfectly homogeneous infinite resistive sheet with conductivity σ. Imagine injecting current I into some point $\mathbf{y}$ and collecting it at the infinitely distant boundary. As long as the material is homogeneous, the current will flow from the injection point in a radially symmetric way with the current density given by

$$\mathbf{j}(\mathbf{x}) = \frac{I(\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2}. \tag{19}$$

Consider what happens if we "short out" the sheet, reducing the resistivity to zero in some region M. This short will break the radial symmetry of the system and change the current flow pattern. Intuitively, we expect that more current will flow in the direction of M. However, this intuition can be incorrect in some cases. We now describe a shape M with the property that shorting out the sheet in M causes current to flow away from M.

The counterexample geometry is as follows. We inject current at some point $\mathbf{y} = (0, z)$ in the upper half plane and short out the sheet along a semicircle centered at the origin and located in the lower half plane [see Fig. 8(a)]. If z is small, shorting out the sheet along the semicircle increases the current flow in the positive z direction in the vicinity of $\mathbf{y}$.

One way to derive this counterintuitive behavior is to note that the current flow can be exactly mapped onto the original electrostatics problem. The current density $\mathbf{j}$ obeys the continuum analogue of Kirchoff's laws,

$$\nabla \cdot \mathbf{j}(\mathbf{x}) = I\delta(\mathbf{x} - \mathbf{y}), \tag{20}$$

$$\nabla \times \left(\frac{\mathbf{j}}{\sigma}\right) = 0, \tag{21}$$

with the boundary conditions

$$\mathbf{j}(\mathbf{x}) \perp \partial M \quad (\mathbf{x} \in \partial M), \tag{22}$$

$$\mathbf{j}(\mathbf{x}) = 0 \quad (\mathbf{x} \to \infty), \tag{23}$$

$$\int_{\partial M} \mathbf{n} \cdot \mathbf{j}(\mathbf{x}) d\mathbf{x} = 0. \tag{24}$$

The first boundary condition comes from the vanishing resistivity in the region M, and the third boundary condition comes from current conservation. These equations are identical to the equations obeyed by the electric field $\mathbf{E}$ in the point charge-metallic object electrostatics problem. But we know that in the charge-semicircle electrostatics problem, the metal semicircle generates a repulsive electric field near the point charge when z is small. Translating this repulsive electric field into current flow language, we conclude that shorting out the semicircle must increase the current flow in the positive z direction, in the vicinity of $\mathbf{y}$.

It is interesting to consider the opposite question as well: how does the current flow change if we cut a hole in the sheet in some region M, effectively making the resistivity infinite there? We expect that this hole will decrease the amount of current flowing toward M. Surprisingly, for some shapes of M, the flow toward M may increase instead.

The counterexample geometry is to inject current at some point $\mathbf{y} = (0, z)$ in the upper half plane and to cut the sheet along two line segments in the lower half plane, which are symmetrical with respect to the vertical axis and which have the property that their extensions pass through the origin [see Fig. 8(b)]. If z is small, the effect of making these cuts is to increase the current flow in the negative z direction, at least in the vicinity of $\mathbf{y}$.

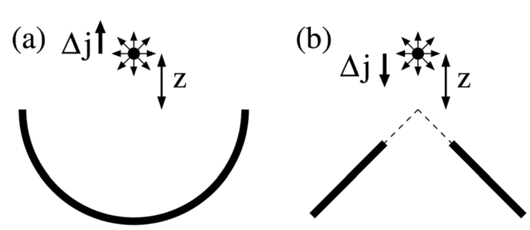

Fig. 8. (a) If current is injected into a homogeneous resistive sheet with conductivity σ, current flows from the injection point in a radially symmetric way. Reducing the resistivity to 0 in a thin semi-circular region causes an increase, $\Delta\mathbf{j}$, in the current flowing away from the semi-circle. (b) Increasing the resistivity to ∞ along two thin line segments intersecting at the origin leads to an increase, $\Delta\mathbf{j}$, in the current flowing toward the lines.

To see this effect, note that in this case, the current density obeys Neumann boundary conditions at ∂M instead of Dirichlet boundary conditions:

$$\mathbf{j}(\mathbf{x}) \parallel \partial M \quad (\mathbf{x} \in \partial M), \tag{25}$$

$$\mathbf{j}(\mathbf{x}) = 0 \quad (\mathbf{x} \to \infty). \tag{26}$$

As a result, this current flow problem maps onto a different kind of electrostatics problem. Instead of the point charge-metallic object system, the analogue system involves a point charge and an object with a dielectric constant that is much smaller than the surrounding medium. Such a geometry is unusual, but could in principle be realized by immersing a point charge and an object with a small dielectric constant in a liquid with a large dielectric constant. Although this electrostatics problem is different from the ones we have considered, we can analyze it in the same way as before. We note that when $z = 0$, the vacuum field lines of the point charge automatically obey the Neumann boundary conditions (26), which means that the electric field lines at $z = 0$ are the same as in a vacuum. Hence, the electrostatic energy U at $z = 0$ is the same as at infinite separation: $U(0) = U(\infty)$. Because the force is repulsive at large z (as follows from general arguments similar to the Dirichlet boundary condition case), we conclude that there is an attractive regime at small z. If we convert this argument into current flow language, we deduce that cutting the sheet along the radial lines increases the current flow in the negative z direction in the vicinity of $\mathbf{y}$, when z is small.

VI. CONCLUSION

We have shown that, in certain geometries, a neutral metallic object can repel a point charge. We have also shown that analogues of this geometry-induced repulsion can appear in Casimir systems and current flow problems. These examples demonstrate that geometry alone can reverse the sign of electrostatic and Casimir forces and lead to surprising behavior in many other systems. We expect that analogues of this effect can appear in almost any physical system governed by Laplace-like equations, from superconductor-magnet systems to (idealized) fluid flow problems.

One direction for future research would be to investigate to what extent these counterexamples are special. For example, are all shapes that repel a point charge similar to the hemisphere geometry discussed here or are there completely different kinds of geometries with this property? More specifically, is it possible to achieve repulsion with a convex metallic object? We can ask similar questions about Casimir repulsion. There are many open questions here, and we have only just begun to understand these counterintuitive geometric effects.

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