Six easy roads to the Planck scale

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We give six arguments that the Planck scale should be viewed as a fundamental minimum or boundary for the classical concept of spacetime, beyond which quantum effects cannot be neglected and the basic nature of spacetime must be reconsidered. The arguments are elementary, heuristic, and plausible and as much as possible rely on only general principles of quantum theory and gravity theory. The main goal of the paper is to give physics students and nonspecialists an awareness and appreciation of the Planck scale and the role it should play in present and future theories of quantum spacetime and quantum gravity.

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I. INTRODUCTION

Max Planck noted in 1899 the existence of a system of units based on the three fundamental constants G, c, and h.[1,2] Their presently accepted values are $G=6.67428(67)\times 10^{-11}\ \mathrm{m^3/kg\,s^2}$, $c=2.99792458\times 10^{8}\ \mathrm{m/s}$, and $h=6.62606896(33)\times 10^{-34}\ \mathrm{kg\,m^2/s}$. These constants are dimensionally independent in the sense that no combination is dimensionless and a length, a time, and a mass may be constructed from them. Specifically, with $\hbar \equiv h/2\pi = 1.05\times 10^{-34}$ J s in preference to h, the Planck scale is

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6\times 10^{-35}\ \mathrm{m}$$

$$T_P = \sqrt{\frac{\hbar G}{c^5}} = 0.54\times 10^{-43}\ \mathrm{s}, \tag{1a}$$

$$M_P = \sqrt{\frac{\hbar c}{G}} = 2.2\times 10^{-8}\ \mathrm{kg} \quad \text{or}$$

$$E_P = M_P c^2 = 1.2\times 10^{19}\ \mathrm{GeV}. \tag{1b}$$

The Planck energy is conveniently expressed in GeV.[3]

The Planck length is very far removed from the human scale of about a meter. We humans are much closer in order of magnitude to the scale of the universe, 10^{26} m, than to the Planck length! The Planck mass is about that of a small grain of sand. However it is much larger than the masses of nucleons and the particles presently studied in high energy physics experiments. Such experiments involve energies only of order 10^3 GeV, and even the highest energy cosmic rays detected to date, about 10^{12} GeV, are far below the Planck energy.

The presence of $\hbar$ in the Planck units in Eqs. (1) indicates that the Planck scale is associated with quantum effects. The presence of c indicates that it is associated with spacetime, and the appearance of G indicates that it is associated with gravity. We therefore expect that the scale is characteristic of quantum spacetime or quantum gravity, which is the present wisdom.[2,3]

We wish to show in this paper that the Planck scale is the boundary of validity of our present standard theories of gravity and quanta. We travel six roads to the boundary using arguments based on thought experiments. It is beyond the scope of the paper to cross the boundary and discuss current efforts toward theories of quantum gravity and spacetime, but we briefly mention a few such efforts in Sec. IX and refer the reader to several useful references.[4–7]

Observational confirmation of Planck scale effects is highly problematic.[3,8–10] We cannot expect to do accelerator experiments at the Planck energy in the foreseeable future, but there are indirect possibilities. One involves the radiation predicted by Hawking to be emitted from black holes if they are small enough to have a Hawking temperature above that of the cosmic background radiation. In the final stages of black hole evaporation, the Hawking radiation should have about the Planck energy.[11] Hawking radiation has not yet been observed, although many theorists believe it must exist. Another possibility is to analyze the radiation from very distant gamma ray bursters, which has been en route for about 10^{10} years.[12] Speculations abound on the effect of Planck scale spacetime granularity on the propagation of such radiation.[13] Also, in the earliest stages of cosmological inflation, quantum gravity effects might have been large enough to leave an imprint via primordial gravitational radiation on the details of the cosmic microwave background radiation.[14]

Lacking real experiments we consider thought (Gedanken) experiments in this paper. We give plausible heuristic arguments why the Planck length should be a sort of fundamental minimum—either a minimum physically meaningful length or the length at which spacetime displays inescapable quantum properties, that is, the classical spacetime continuum concept loses validity. Specifically the six thought experiments involve viewing a particle with a microscope, measuring a spatial distance with a light pulse, squeezing a system into a very small volume, observing the energy in a small volume, measuring the energy density of the gravitational field, and determining the energy at which gravitational forces become comparable to electromagnetic forces. The analyses require a minimal knowledge of quantum theory and some basic ideas of general relativity and black holes, which we will discuss in Sec. II. Some background in classical physics, including special relativity, is also assumed.

We rely as little as possible on present theory both because we desire mathematical and pedagogical simplicity and because the general principles we use are most likely to survive the vagaries of theoretical fashion. We hope that the discussions are thereby accessible to physics undergraduates and nonspecialists.

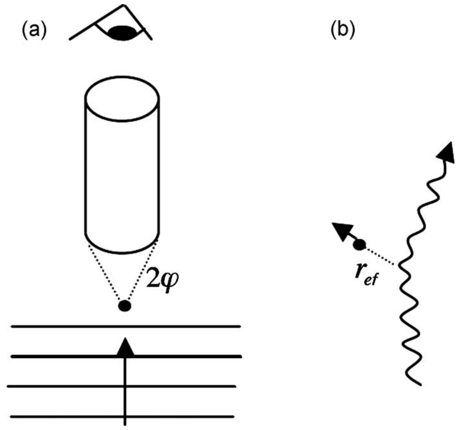

Fig. 1. (a) A particle is illuminated from below by light of wavelength λ, which scatters into the microscope whose objective lens subtends an angle of 2φ. (b) The particle nature of the scattering is emphasized, with an effective separation r_{eff} shown.

II. COMMENTS ON BASIC IDEAS OF QUANTUM THEORY AND GRAVITY

In this section we review the quantum theory and gravitational theory needed in later sections. Those familiar with quantum theory and general relativity may proceed to the first road in Sec. III.

The first aspect of quantum theory that we recall is the quantization of light. According to Planck and Einstein, light of frequency ν and wavelength $\lambda = c/\nu$ can be emitted and absorbed only in multiples of the energy

$$E = h\nu. \tag{2}$$

Thus we may think of light as a rain of photons, each with $E = h\nu$.[15,16] According to Einstein's relation for mass-energy equivalence, $E = mc^2$, a photon should interact gravitationally as if it has an effective mass

$$M_{\text{eff}} = h\nu/c^2. \tag{3}$$

For example, a single photon captured in a mirrored cavity increases the effective mass of the cavity according to Eq. (3).

Next we recall a standard heuristic derivation of the Heisenberg uncertainty principle.[15] In Fig. 1 we show the Heisenberg microscope thought experiment, where by using the microscope we view a particle with light entering from the bottom of the page. It is well known from wave optics (and also intuitively) that the position of the particle can be determined to an accuracy of about the wavelength λ of the light used, or a bit more precisely the uncertainty in position is given by

$$\Delta x \cong \lambda/\sin \varphi. \tag{4}$$

According to classical physics, we can determine the position as accurately as desired by using very short wavelength light of arbitrarily low intensity so as not to disturb the particle by the electric field of the light wave. The quantization of light as photons with energy $h\nu$ prevents this determination because the intensity cannot be made arbitrarily low. A single photon scattering from the particle and into the microscope (at an angle less than φ) will impart a momentum of order $\Delta p \cong p \sin \varphi = (h/\lambda)\sin \varphi$ to the particle so that Eq. (4) implies

$$\Delta x \Delta p \cong h \approx \hbar, \tag{5}$$

which is the Heisenberg uncertainty principle. (In our rough estimates, we do not distinguish between h and $\hbar$, thus taking $2\pi \approx 1$. Some physicists informally refer to these units as "Feynman units.")

The uncertainty principle forces us to consider the position and momentum of the particle to be imprecise or "fuzzy" so that the particle occupies a region of at least $\hbar$ in phase space (x,p). Thus we cannot speak of the trajectory of a quantum mechanical particle but must take account of the fuzziness of such a particle in our description, that is, in terms of a wave function or probability amplitude.

Most quantum mechanics textbooks also give a derivation of the uncertainty principle from the commutation relation for the position and momentum operators, and it is also readily obtained from the fact that the wave functions in position and momentum space are Fourier transforms of each other.[16] These more formal derivations consider a somewhat different interpretation of the uncertainty principle than the one we use here. Such questions of interpretation have been of interest since the earliest days of quantum mechanics (typified by the debates between Bohr and Einstein) but are beyond the scope of this paper. We refer the reader interested in the epistemology and interpretation of quantum mechanics, especially the uncertainty principle, to Ref. 16.

There is an energy-time analog of the uncertainty principle Eq. (5), but it has a somewhat different meaning. Consider a wave of frequency approximately ν and duration T, which thus consists of $N \approx \nu T$ cycles. The finite duration of the wave means that its leading and trailing edges will be distorted somewhat from sinusoidal so that N will not be precisely well defined and measurable and will have an uncertainty of order $\Delta N \approx 1$. This uncertainty in N implies an uncertainty in the frequency given by $\Delta \nu T \approx \Delta N \approx 1$. (This relation is well known in many fields such as optics and electrical engineering, where it relates band-width and pulse-width.[17] The time T is subject to several somewhat different interpretations. The relation is also easy to derive more formally by calculating the Fourier transform of a finite nearly monochromatic wave train, which is its frequency spectrum.)

Because the energy of a photon of light is given by $E = h\nu$, we see that its energy can be measured only to an accuracy given by

$$(\Delta E)(T) \approx h \approx \hbar, \tag{6}$$

where ΔE is the absolute value of the energy uncertainty. (The same relation holds by similar reasoning for many other quantum systems.)

Equation (6) formally resembles the uncertainty principle in Eq. (5), but unlike the position of a particle, time is not an observable in quantum mechanics, and therefore Eq. (6) has a different meaning: T is the characteristic time of the system (for example, pulse-width) and not an uncertainty in a time measurement.[18] For example, the light emitted by an atom is only approximately monochromatic, with an energy uncertainty given by $\Delta E \approx \hbar/T$, where T is the lifetime of the excited atomic state.

The quantization relation, Eq. (2), and the uncertainty relations, Eqs. (5) and (6), are sufficient for our later analyses, and we need not discuss quantum mechanics in more detail or quantum field theory.

We next consider general relativistic gravity and black holes. The line element or metric of special relativity gives the spacetime distance between nearby events.[19] It is usually expressed in Lorentz coordinates $(ct, \vec{x})$ as

$$ds^2 = (cdt)^2 - d\vec{x}^2. \tag{7}$$

In general relativity, gravity is described by allowing spacetime to be warped or distorted, or, more technically correct, curved. Coordinates in general relativity label the points in spacetime and do not by themselves give physical distances; for that we need a metric, which relates coordinate intervals to physical distance intervals.[3,20–22] For a weak gravitational field and slowly moving bodies, the metric is approximately given by the Newtonian limit

$$ds^2 = (1 + 2\phi/c^2)(cdt)^2 - (1 - 2\phi/c^2)(d\vec{x})^2. \tag{8}$$

Here ϕ is the Newtonian potential, with the dimensionless quantity ϕ/c^2 assumed to be small and to go to zero asymptotically at large distances from the source, such as a point mass M where the potential is $\phi = -GM/r$. Equation (8) means that the proper time, or physical clock time, between two events at the same space position and separated only by a time coordinate interval cdt is given by

$$ds = cdt\sqrt{1 + 2\phi/c^2} \quad \text{(proper time separation)}, \tag{9a}$$

and the physical distance between two events separated only by a space coordinate interval dx is

$$dx\sqrt{1 - 2\phi/c^2} \quad \text{(space separation)}, \tag{9b}$$

and similarly for the y and z directions. (Note that the coordinates in Eq. (8) are well chosen for describing an inertial system in which the notion of points separated only by a spatial coordinate interval is clear.)

The Newtonian limit Eq. (8) is a good approximation in many cases. For example, in the solar system the Newtonian potential is greatest at the surface of the Sun, where $2\phi/c^2 \approx 10^{-6}$. Thus spacetime in the solar system is extremely close to that of special relativity or flat. Because we are interested only in order of magnitude estimates, we will make free use of the Newtonian limit Eq. (8) as an approximation.[20–23]

The exact Schwarzschild metric, which describes a black hole or the exterior of any spherically symmetric body, is given in the same coordinates as Eq. (8) by

$$ds^2 = \frac{(1 - GM/2c^2r)^2}{(1 + GM/2c^2r)^2}(cdt)^2 - (1 + GM/2c^2r)^4 d\vec{x}^2. \tag{10}$$

This expression is correct only in matter-free space outside the body.[24] At large distances from the body, it is

$$\begin{aligned} ds^2 &\approx (1 - 2GM/c^2r)(cdt)^2 \\ &\quad - (1 + 2GM/c^2r)d\vec{x}^2, \quad GM/c^2r \ll 1, \end{aligned} \tag{11}$$

which agrees with the Newtonian limit Eq. (10). If the body is small enough so that $r = GM/2c^2$ lies outside of it, then the coefficient of the time term in Eq. (10) vanishes, which means that the proper time ds is zero, and a clock at that position would appear to a distant observer to stop. This radius defines the surface of the black hole. Light cannot escape from the surface because it undergoes a redshift to zero frequency. A black hole of typical stellar mass is about 1 km in radius.[3]

A minor caveat concerning Eq. (10) is in order. The coordinate r used in Eq. (10) is called the isotropic radial coordinate and is not the same as that usually used for the Schwarzschild metric, the Schwarzschild radial coordinate. We will not use the Schwarzschild radial coordinate here (or define it) but note only that it is asymptotically equal to the isotropic coordinate r in Eq. (10) and differs from it at the black hole surface by a factor of 4. Thus, the black hole surface is at radius $2GM/c^2$ in the Schwarzschild coordinate, and $2GM/c^2$ is widely known as the Schwarzschild radius.[24]

It is generally believed that if enough mass M is squeezed into a roughly spherical volume of size $r \approx GM/c^2$, then it must collapse to form a black hole, regardless of internal pressure or other opposing forces. However, if the mass is needle or pancake shaped, the question of collapse is not yet clearly settled.[25,26]

In brief, the lesson to take away from this section is that spacetime distortion is a measure of the gravitational field. Specifically, because the line element gives the physical distance between nearby spacetime points or events, such distances are given roughly by Eqs. (9). This distance corresponds to a fractional distortion given by

$$\text{spacetime fractional distortion} \approx \frac{|\phi|}{c^2}. \tag{12a}$$

Alternatively, from Eq. (10), we can say that for a roughly spherical system of size ℓ, which contains a mass M, the fractional distortion is of order

$$\text{Spacetime fractional distortion} \approx \frac{GM}{\ell c^2}. \tag{12b}$$

Equations (12) at least roughly hold even for rather strong gravitational fields.

III. THE GENERALIZED UNCERTAINTY PRINCIPLE

Our first road to the Planck scale is based on the same thought experiment as the Heisenberg microscope used to obtain the uncertainty principle in Sec. II and shown in Fig. 1 and includes the effects of gravity to obtain a generalization of the uncertainty principle.[27,28] According to the uncertainty principle in Eq. (3), there is no limit on the precision with which we may measure a particle position if we allow a large uncertainty in momentum, as would result from using arbitrarily short wavelength light. But Eq. (3) does not take into account the gravitational effects of even a single photon. As noted in Sec. II, a photon has energy $h\nu$ and thus an effective mass $M_{\mathrm{eff}} = h\nu/c^2 = h/c\lambda$, which will exert a gravitational force on the particle. This force will accelerate the particle, making the position of the already fuzzy particle even fuzzier. By using classical Newtonian mechanics, we can estimate the acceleration and position change due to gravity as roughly

$$\Delta a_g \approx GM_{\mathrm{eff}}/r_{\mathrm{eff}}^2 = G(h/\lambda c)/r_{\mathrm{eff}}^2, \tag{13a}$$

$$\Delta x_g \approx \Delta a_g t_{\mathrm{eff}}^2 \approx G(h/\lambda c)(t_{\mathrm{eff}}^2/r_{\mathrm{eff}}^2), \tag{13b}$$

where r_{eff} and t_{eff} denote an effective average distance and time for the interaction. The only characteristic velocity of the system is the photon velocity c, so we take $r_{\mathrm{eff}}/t_{\mathrm{eff}} \approx c$ and obtain for the gravitational contribution to the uncertainty

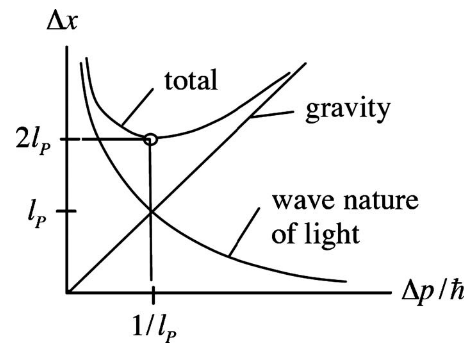

Fig. 2. The position uncertainty due to the standard uncertainty principle and the additional gravitational effect embodied in the generalized uncertainty principle. The minimum uncertainty occurs at about twice the Planck length.

$$\Delta x_g \approx Gh/\lambda c^3 \approx (G\hbar/c^3)/\lambda = \ell_P^2/\lambda. \tag{14}$$

According to Eq. (5), the position uncertainty neglecting gravity is about $\Delta x \approx \hbar/\Delta p$. We add the gravitational contribution in Eq. (14) to obtain a generalized uncertainty principle

$$\Delta x \approx \left(\frac{\hbar}{\Delta p}\right) + \ell_P^2\left(\frac{\Delta p}{\hbar}\right). \tag{15}$$

This same expression for the generalized uncertainty principle has been obtained in numerous ways, ranging in sophistication from our naive Newtonian approach to several versions of string theory.[27] It thus appears to be a general result of combining quantum theory with gravity and may be correct.[28]

The generalized uncertainty principle should be considered to be a rough estimate for the position uncertainty, with the coefficient of $\Delta p/\hbar$ in the second term in Eq. (15) being only of the order of the square of the Planck length, and the entire expression being valid only in order of magnitude as we approach the Planck scale from above. For example, we could just as well have factors of 2π, 10, $1/\alpha=137$, etc., multiplying ℓ_P^2 in Eq. (15). Moreover, we have added the uncertainties due to the standard uncertainty principle and the gravitational interaction linearly, and we could equally well have taken the root mean square. This difference would make little difference in our conclusions.

From Eq. (15) we see that the position uncertainty of a particle has a minimum at $\hbar/\Delta p=\ell_P$ and is about

$$\Delta x_{\text{min}} \approx 2\ell_P, \tag{16}$$

as shown in Fig. 2. This minimum position uncertainty corresponds to a photon of wavelength of ℓ_P and energy E_P.

Because we cannot measure a particle position more accurately than the Planck length, result (16) suggests that from an operational perspective, the Planck length may represent a minimum physically meaningful distance. As such, we may be skeptical that theories based on arbitrarily short distances, such as classical or quantum field theory, really make sense from an operational point of view. That is, we may question whether the idea of a differentiable manifold provides a useful model at the Planck scale. It may be that the concept of spacetime at such a scale is not useful, as we will discuss in Sec. IX.[3,29]

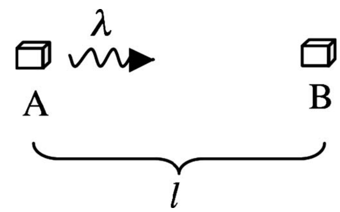

Fig. 3. A light pulse is sent from A and reflected back from B. Its energy causes a distortion of the spacetime between A and B and hence affects the length ℓ.

IV. LIGHT RANGING

The next argument uses a thought experiment that is particularly simple conceptually and has the virtue that length is defined via light travel time, which is how the meter is defined.[30]

Figure 3 shows the experimental arrangement, which we refer to by the generic name light ranging, in analogy with laser ranging. We send a pulse of light with wavelength λ from a position labeled A to one labeled B, where it is reflected back to A, and measure the travel time with a macroscopic clock visible from A. Because light is a wave, we cannot ask that the pulse front be much more accurately determinable than about λ, so there is an uncertainty of at least about $\Delta\ell_w \approx \lambda$ in our measurement of the length ℓ.

If nature were actually classical, we could use light of arbitrarily short wavelength and arbitrarily low energy so as not to disturb the system and thereby measure the distance to arbitrarily high accuracy. In the real world we cannot use arbitrarily short λ because it would put large amounts of energy and effective mass in the measurement region, even if we use only a single photon.[3] According to our comments in Sec. II, the energy of the light will distort the spacetime geometry and thus change the length ℓ by a fractional amount $|\phi|/c^2$ according to Eqs. (12). We may estimate the Newtonian potential due to the photon, which is somewhere in the interval ℓ, to be about

$$\phi \approx \frac{GM_{\text{eff}}}{\ell} \approx \frac{G(E/c^2)}{\ell} \approx \frac{Gh\nu}{c^2\ell} \approx \frac{G\hbar}{c\ell\lambda}, \tag{17}$$

and hence the spatial distortion is about

$$\Delta\ell_g \approx \ell(\phi/c^2) \approx (G\hbar/c^3)/\lambda = \ell_P^2/\lambda. \tag{18}$$

In Fig. 3 the letter B labels a point in space, but we could instead take it to be a small body in free fall, which would move during the measurement (actually only during the return trip of the light pulse), and this movement would also affect the measurement. We have estimated this motion in Sec. III, and it is given in Eq. (14) by

$$\Delta x_g \approx \ell_P^2/\lambda. \tag{19}$$

That is, the space distortion in Eq. (18) and the motion in Eq. (19) are comparable, and to our desired accuracy, we simply write for either effect $\Delta\ell_g \approx \ell_P^2/\lambda$.

Because we know only the photon position to be somewhere in ℓ, we interpret Δx_g in Eq. (19) as an additional uncertainty due to gravity. We add it to the uncertainty due to the wave nature of the light to obtain

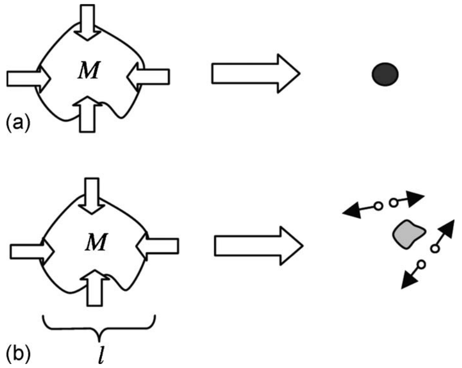

Fig. 4. The shrinking of a volume containing mass M is limited by gravitational collapse to a black hole in (a) and by the creation of particle antiparticle pairs in (b).

$$\Delta\ell \approx \Delta\ell_w + \Delta\ell_g \approx \lambda + \ell_P^2/\lambda. \tag{20}$$

The expression in Eq. (20) for the total uncertainty has a minimum at $\lambda = \ell_P$, where it is equal to $\Delta\ell \approx 2\ell_P$. Thus we conclude that the best we can do in measuring a distance using light ranging is about the Planck length. The energy and gravitational field of the photon prevent us from doing better.

We have assumed implicitly that we have access to a perfect classical clock for timing the light pulse. If the clock is assumed to be a small quantum object, there will be a further contribution to the uncertainty due to the spread of the position wave function of the clock during the travel of the light pulse. Some authors suggest that such a quantum clock should be used in the thought experiment and arrive at a larger uncertainty estimate for light ranging, one involving the size of the system ℓ.[9,31] Other authors point out that such a quantum clock may not be appropriate because it could suffer decoherence and behave classically.[10] Predictions of this nature may be testable with laser interferometers constructed as gravitational wave detectors.[32]

V. SHRINKING A VOLUME

In this thought experiment, we shrink a volume containing a mass M as much as possible until we are prevented by the theory from continuing. We assume the volume is intrinsically three-dimensional, about ℓ in all of its spatial dimensions, as shown in Fig. 4. A difficulty occurs due to gravity when the size approaches the Schwarzschild radius,

$$\ell \approx GM/c^2. \tag{21}$$

The system may then collapse to form a black hole as discussed in Sec. II and cannot be made smaller. There is no lower limit to this size if we choose a small mass M.

A different difficulty occurs due to quantum effects. From the uncertainty principle, the uncertainty in the momentum of the material in the volume is at least of order $\Delta p \approx \hbar/\ell$. Because the energy in the volume is given by $E^2 = M^2c^4 + p^2c^2$, the uncertainty in the energy is roughly

$$\Delta E \approx c\Delta p \approx \hbar c/\ell. \tag{22}$$

If ℓ is made so small that this energy uncertainty increases to about $2Mc^2$, then pairs of particles can be created and appear in the region around the mass M, as shown in Fig. 4.[33] The localization is thereby ruined, and the volume cannot shrink further. This limit occurs when the energy and size are about

$$Mc^2 \approx \Delta E \approx \hbar c/\ell, \quad \ell \approx \hbar/Mc. \tag{23}$$

The quantity $\hbar/Mc$ is known as the Compton radius or Compton wavelength for the mass M.

Our inability to localize a single particle to better than its Compton radius is well known in particle physics. One fundamental reason that quantum field theory is used in particle physics is that it can describe the creation and annihilation of particles, whereas a single particle wave function cannot.[34]

We now have two complementary minimum sizes for the volume containing a mass M: The Schwarzschild radius dictated by gravity is proportional to M, and the Compton radius dictated by quantum mechanics is inversely proportional to M. The overall minimum occurs when the two are equal, which occurs for

$$\ell \approx \hbar/Mc \approx GM/c^2, \quad M^2 \approx \hbar c/G \equiv M_p^2, \tag{24a}$$

$$\ell \approx \hbar/M_pc \approx \sqrt{\hbar G/c^3} = \ell_P. \tag{24b}$$

Thus the combination of gravity and quantum effects creates insurmountable difficulties if we attempt to shrink a volume to smaller than the Planck size.

A minor caveat should be repeated here: We have assumed that the volume is effectively three-dimensional in that all of its dimensions are roughly comparable. If one of the dimensions is much larger or much smaller than the others, the region is effectively one- or two-dimensional, and the question of gravitational collapse is less clear, as noted in Sec. II.[25,26]

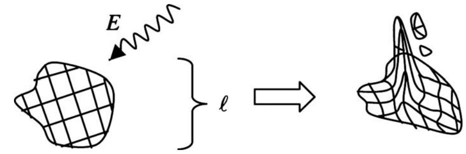

Fig. 5. A region of space of size ℓ to be measured in time ℓ/c. As the size approaches the Planck length, there can occur wild variations in the geometry, including such things as black holes and wormholes.

VI. MEASURING PROPERTIES OF A SMALL VOLUME

For this generic thought experiment, we use a quantum probe, such as a light pulse, to measure the size, energy content, or other physical properties of a volume of characteristic size ℓ, as shown in Fig. 5. Such properties may in general fluctuate significantly in the time it takes light to cross the volume, so we would naturally want to do the measurement within that time, $T \approx \ell/c$. To do so we need a probe with frequency greater than c/ℓ and energy greater than about $E \approx \hbar(c/\ell)$.

But there is a limit to how much probe energy E or effective mass $M_{\text{eff}} \approx E/c^2 \approx \hbar/\ell c$ can be packed into a region of size ℓ, as we discussed in Sec. II. According to Eqs. (12) the fractional distance uncertainty in the volume containing such an effective mass is about

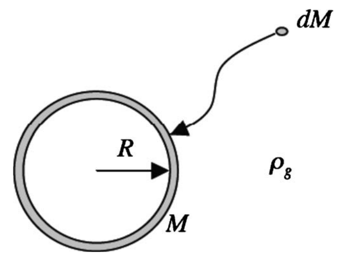

Fig. 6. Potential energy of many small dM elements reappears as energy density ρ_g of the gravitational field.

$$\frac{\Delta \ell}{\ell} \approx \frac{|\phi|}{c^2} \approx \frac{1}{c^2}\left(\frac{GM_{\text{eff}}}{\ell}\right) = \frac{G}{c^2\ell}\left(\frac{\hbar}{\ell c}\right) = \frac{G\hbar}{c^3\ell^2} = \frac{\ell p^2}{\ell^2}. \quad (25)$$

If ℓ is made so small that this ratio approaches 1, then the geometry becomes greatly distorted and the measurement fails, which occurs at $\ell \approx \ell_P$.

Another way to see the limit effect is to note that the effective mass $M_{\text{eff}} \approx \hbar / c\ell$ injected into the region by the probe can induce gravitational collapse to form a black hole (as noted in Sec. V) when the region size approaches the Schwarzschild radius of about GM_{eff}/c^2, which occurs for

$$l \approx \frac{GM_{\text{eff}}}{c^2} \approx \frac{G\hbar}{c^3\ell} \quad (26)$$

or $\ell \approx \sqrt{G\hbar / c^3} \approx \ell_P$.

We thus conclude that any attempt to measure physical properties in a region of about the Planck length involves so much energy that large fluctuations in the geometry must occur, including the formation of black holes and probably more exotic objects such as wormholes.[35] Such wild variations in the geometry were dubbed spacetime foam by Wheeler, and the phrase has become popular to express the supposed chaotic nature of geometry at the Planck scale.[36]

VII. ENERGY DENSITY OF GRAVITATIONAL FIELD

This argument is based on the uncertainty in the energy density of the gravitational field and field fluctuations that correspond to the uncertainty. Algebraically it resembles somewhat the argument in Sec. VI but has a different conceptual basis.

We first obtain an expression for the energy density of the gravitational field in Newtonian theory. Consider assembling a spherical shell of radius R and mass M by moving small masses from infinity to the surface, as shown in Fig. 6. From Newtonian theory, the energy to move a small mass dM to the surface is

$$dE = -GMdM/R, \quad (27)$$

and the total energy for the assembly is the integral of this energy over the mass, which is

$$E = -GM^2/2R. \quad (28)$$

This binding energy may be viewed as energy in the gravitational field and is negative because gravity is attractive. To obtain a general expression for the energy density, we assume it is proportional to the square of the gravitational field $\vec{g} = -\nabla\phi$. That is, we set

$$\rho_g = \lambda(\nabla\phi)^2. \quad (29)$$

The expression in Eqs. (29) is analogous to the energy density of the electric field—except for the opposite sign. The proportionality constant λ can be determined by integrating ρ_g in Eqs. (29) over the volume between R and infinity in Fig. 6 to obtain the total field energy. If we equate this energy expression with the total binding energy given by Eq. (28), we obtain

$$\lambda(4\pi G^2M^2/R) = -GM^2/2R \quad (30)$$

and $\lambda = -1/8\pi G$. Thus the energy density of the Newtonian gravitational field may be written as

$$\rho_g = -(\nabla\phi)^2/8\pi G. \quad (31)$$

In general relativity the problem of defining the energy density of the gravitational field is subtle and more complex than in Newtonian theory.[37] The difficulty is due to the need for general covariance, that is, the transformation character of the gravitational energy, and also to the nonlinearity of the field equations. This nonlinearity is related to the fact that gravity carries energy and is thus a source of more gravity. In this sense the gravitational field differs fundamentally from the electric field, which does not carry charge and thus is not the source of more electric field. For our purpose we thus will content ourselves with the rough estimate for the energy density given in Eq. (31) and eschew the largely unsolved general problem.

We now consider a spatial region of size ℓ that is nominally empty and free of gravity, except for fluctuations allowed by the energy-time uncertainty relation in Eq. (6). As in Sec. VI we measure the gravitational energy in the region in the time ℓ / c, with accuracy limited to $\Delta E \approx \hbar c / \ell$. We thus cannot verify that the region is truly free of gravity but only that the gravitational energy in the region is no more than about $\Delta E \approx \hbar c / \ell$. From Eq. (31) this limit implies the following limiting relation for the Newtonian potential field:

$$\ell^3(\nabla\phi)^2/8\pi G \approx \hbar c/\ell. \quad (32)$$

As a rough estimate $(\nabla\phi)^2 \approx (\Delta\phi / \ell)^2$, where $\Delta\phi$ is the uncertainty or fluctuation in the nominally zero Newtonian potential field. Hence, from Eq. (32) we obtain

$$\Delta\phi \approx \sqrt{\hbar c G}/\ell. \quad (33)$$

This fluctuation corresponds roughly to a fractional spacetime distortion given by Eqs. (12),

$$\Delta\ell/\ell \approx \Delta\phi/c^2 \approx \sqrt{\hbar G/c^3}/\ell \quad (34)$$

and $\Delta\ell \approx \sqrt{\hbar G / c^3} = \ell_P$. That is, the allowed nonzero value of the energy density of the gravitational field corresponds to Newtonian potential fluctuations and thus metric and distance fluctuations; the distance fluctuations are about the Planck length.

VIII. EQUALITY OF GRAVITY AND ELECTRIC FORCES

Our final argument characterizes the Planck scale in terms of the mass or energy at which gravitational effects become comparable to electromagnetic effects and thus cannot be ignored in particle theory. The argument is simple to remember and provides a good mnemonic for quickly deriving the Planck mass.

In most situations we encounter gravity as an extremely weak force. For example, the gravitational force between electron and proton in a hydrogen atom is roughly 40 orders of magnitude less than the electric force and can be ignored.[3] If we instead consider two objects of very large mass M (or rest energy) with electron charge e, then the gravitational and electric forces become equal when

$$\frac{GM^2}{r^2} \cong \frac{e^2}{r^2} \tag{35}$$

or $M^2 \cong e^2/G$. The fine structure constant is defined as $\alpha \equiv e^2/\hbar c \cong 1/137$, and thus we may express Eq. (35) in terms of the Planck mass M_P as

$$M^2 \cong \alpha \frac{\hbar c}{G} = \alpha M_P^2 \text{ and } M \cong \sqrt{\alpha} M_P \cong \frac{M_P}{12}. \tag{36}$$

That is, equality occurs about an order of magnitude from the Planck mass, at least in terms of Newtonian gravity. We may plausibly infer that this equality also occurs when charged massive particles scatter at near the Planck energy.[38]

Quantum electrodynamics, which describes the electromagnetic interactions of electrons and other charged particles, ignores gravitational effects. This neglect is not reasonable for energies near the Planck scale. There are some virtual processes in the theory that involve integrals over arbitrarily high energies, which are thus clearly not handled correctly, and most such integrals diverge.[3,32,33] We may therefore hope that a more comprehensive theory that includes gravity might be free of such divergences.

Despite the simplistic nature of this section, it hints at the germ of deep ideas. In the standard model of particle physics the electromagnetic and weak forces are unified into the electroweak force, which has been very successful in predicting experimental results. At low interaction energies the weak and electromagnetic forces differ greatly, but at an energy above a few thousand GeV, they become comparable.[38,39] They may be viewed as different aspects of a single force rather than as fundamentally different forces. Similarly the strong force is widely believed to become comparable and similarly unified with the electroweak force in some grand unified theory at energies of about 10^{16} GeV, only a few orders of magnitude below the Planck energy.[39] Roughly speaking, all the fundamental forces of nature are believed to become comparable near the Planck scale.

IX. SUMMARY AND FURTHER COMMENTS

We have shown that the Planck scale represents a boundary when we attempt to apply our present ideas of quantum theory, gravity, and spacetime on a small scale. To go beyond that boundary, new ideas are needed.

There is much speculation on such new ideas. We will mention only three of many such efforts and refer the reader to Refs. 4–7. The first effort involves perturbative quantum gravity studied for many years by many authors, notably Feynman and Weinberg. In perturbative quantum gravity, the flat space of special relativity is taken to be a close approximation to the correct geometry, and deviations from it are treated in the same way as more ordinary fields such as those in electromagnetism. As in quantum electrodynamics Feynman diagrams may be used to describe the interactions between particles and the quanta of the gravitational field, called gravitons. The theory has the serious drawback that it does not renormalize in the same way as quantum electrodynamics and contains an infinite number of parameters and graviton interactions. More importantly it does not truly address the quantum nature of spacetime.

The best-known effort involves superstring theory, or simply string theory, in which the point particles assumed in quantum field theories are replaced by one-dimensional strings of about the Planck size. String theory purports to describe all particles and interactions and has been studied intensively for decades. It is consistent with gravitational theory because it accommodates a particle with the properties of the graviton, that is, zero mass and spin 2. There is yet no observational evidence that its basic premise is correct.

Another effort, which is widely called loop quantum gravity, recasts the mathematics of general relativity in a way such that the fundamental object is not the metric but an object called an affine connection, which is analogous to the gauge potentials describing other nongravitational fields, such as the vector potential of the electromagnetic field. In loop quantum gravity, areas and volumes are quantized, and the theory has other attractive features.

Various authors visualize spacetime as a boiling quantum foam of strange geometries such as virtual black holes and wormholes, as a dense bundle of six-dimensional Calabi–Yau manifolds, as a subspace of a more fundamental ten- or eleven-dimensional space, as a lower dimensional holographic projection, as the discrete eigenvalue spectrum of a quantum operator, as a spin network, or as a woven quantum fabric.[6] But none of the many speculative ideas and theories have yet reached a high level of success or general acceptance, and we remain free to consider many possibilities.

Perhaps the oddest possibility is that spacetime at the Planck scale is not truly observable and thus may be an extraneous and sterile concept, much as the luminiferous aether of the 19th century proved to be extraneous after the advent of relativity, thus obviating decades of theoretical speculation.[3] At present it is not clear what might replace our present concept of spacetime at the Planck scale.

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[1]Planck's comments on his unit system preceded his discovery of the correct blackbody radiation law. M. Planck, "Uber irreversible Strahlungsvorgange," Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften zu Berlin **5**, 440–480 (1899). An earlier suggestion for a natural system of units was made by G. C. Stoney in 1874, based on G, c, and e. See J. D. Barrow, *The Constants of Nature* (Vintage Books/Random House, New York, 2004), Chap. 2. Stoney's scale differs conceptually from Planck's but is numerically close due to the coincidence that e is roughly equal to $\sqrt{\hbar c}$ to about an order of magnitude.

[2]A. S. Eddington called attention to the Planck scale in 1918, and J. A. Wheeler later identified it as appropriate to quantum gravity, bringing it to the wider attention of theorists. J. A. Wheeler, "Geons," Phys. Rev. **97**(2), 511–536 (1955). Reprinted in J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962). For up-to-date values of the constants, see ⟨www.nist.gov⟩.

[3] R. J. Adler, "Gravity," in *The New Physics for the Twenty-First Century*, edited by G. Fraser (Cambridge U. P., Cambridge, 2006), Chap. 2.
[4] R. P. Feynman, *Lectures on Gravitation* (California Institute of Technology, Pasadena, CA, 1971), Lecture 3, notes taken by F. Morinigo and W. G. Wagner.
[5] The most well-known candidate is (super)string theory. A nontechnical discussion is given by M. B. Green, "Superstring theory," in *The New Physics for the Twenty-First Century*, edited by G. Fraser (Cambridge U. P., Cambridge, 2006). A technical set of notes is available by G. t'Hooft, "String theory lectures," ⟨www.phys.uu.nl/-thooft/lectures/stringnotes.pdf⟩. A popular discussion including other approaches is by L. Smolin, *Three Roads to Quantum Gravity* (Weidenfeld and Nicolson, London, 2000).
[6] An up-to-date technical reference is *Approaches to Quantum Gravity*, edited by D. Oriti (Cambridge U. P., Cambridge, 2009). In particular, see Chapter 1 by C. Rovelli, "The unfinished revolution," in which he points out that the subject of quantum gravity dates back at least to M. P. Bronstein, "Quantentheories schwacher Gravitationsfelder," Phys. Z. Sowjetunion **9**, 140–157 (1936).
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[8] See A. Akahito, "How to build a Planck-mass accelerator in your solar system," Sci. Am. **260**(4), 112–115 (1989). This article is apparently an April fool's joke by T. Rothman using a pseudonym.
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