Poincaré's proof of the quantum discontinuity of nature

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In his last memoir on mathematical physics, Henri Poincaré presented one of the most profound and compelling proofs of the hypothesis of quanta. This highly original proof, which is actually three separate proofs, is based on first principles and is full of physical insight, mathematical rigor, and elegant simplicity. The memoir is refreshingly uncluttered by some of the conventional, and more abstract concepts, such as temperature and entropy, that Planck and others relied on in their work. Poincaré's analysis is based on an ingenious physical model consisting of long-period resonators interacting with short-period resonators. A unique formulation of statistical mechanics, based on the calculus of probabilities, Fourier's integral, and complex analysis, logically unfolds throughout the memoir. Poincaré invents an "inverse statistical mechanics" that allows him to prove a crucial result that no one had proved before: The hypothesis of quanta is both a sufficient and a *necessary* condition to account for Planck's law of radiation. In a separate, more universal proof, Poincaré proves that the existence of a discontinuity in the motion of a resonator is necessary to explain *any* observed law of radiation. Given the significant impact of Poincaré's memoir on quantum theory and statistical physics, it is surprising that most physicists are not aware of its valuable mathematical and physical ideas. Poincaré's tour de force proofs are presented here in a form suitable for use in a standard course in quantum mechanics, statistical mechanics, or mathematical physics.

I. INTRODUCTION

The year 1900 marked the beginning of the quantum theory. In order to theoretically explain the observed law of blackbody radiation, Max Planck introduced the "hypothesis of quanta."[1] According to this hypothesis, the energy of the radiators of light, or resonators, cannot be continuous, but must vary in discrete amounts, called quanta. Naturally, this bold, anticlassical hypothesis gained acceptance at a very slow rate. Poincaré's memoir was a decisive factor in achieving worldwide acceptance.

Are quanta absolutely necessary? This was the fundamental question that physicists were asking during the years 1900–1912. Planck's analysis demonstrated that the hypothesis of quanta was sufficient to account for Planck's law of radiation, which was the formula that best fit the experimental data. However, this sufficient condition is much weaker than a necessary condition. Planck's sufficiency proof does not preclude the existence of other hypotheses that lead to the same radiation law. Furthermore, one could imagine radiation laws that differ from Planck's law within the experimental errors of observation, and which could be explained classically. Before one undertakes a radical revision of the time-honored classical ideas, it would be wise to first determine with certainty whether or not the observations could be explained by some exotic, yet still classical (continuous) mechanism.

It was Poincaré who went well beyond the sufficiency condition to prove that the hypothesis of quanta was absolutely necessary to explain Planck's law of radiation. In a separate, more universal proof, Poincaré completely dissociated himself from the specificity of Planck's experimental law. He proved that there cannot exist a classical theory, based on continuity, that could account for Planck's, or any other experimental law of radiation, as long as the total radiation energy is finite. This shattered all hopes of a classical explanation. A quantum discontinuity is the one, and only one, dynamical mechanism that could explain the observed phenomena. Thanks to Poincaré, quanta were inescapable.

Upon first encountering the strange quantum concept, a curious physics student of the 1990s is prone to ask the same basic question as a classical physicist of the 1900s: "But are quanta, or energy discontinuities, absolutely necessary?" Poincaré's necessity proof and universality proof can provide a definitive answer to this question, which can be especially convincing and comforting to any quantum skeptics in the class. Poincaré's proofs, as presented here, can add a substantial amount of rigor, along with new physical and mathematical ideas, to the conventional approach of introducing quanta via Planck's sufficiency hypothesis.

In Sec. II, relevant historical background information is provided. Section III discusses Poincaré's physical model of interacting resonators, and his basic statistical mechanical formalism of the energy partition. In Sec. IV, Poincaré's three main proofs are given. Section V discusses the impact of the memoir and Sec. VI concludes.

II. BACKGROUND TO THE MEMOIR

A. Poincaré's first encounter with quanta

As a mathematician, physicist, and philosopher, Henri Poincaré (29 April 1854–17 July 1912) was a virtuoso. Acknowledged as the greatest mathematician at the turn of the 20th century, he was the last "universalist" who excelled in virtually all fields of mathematics, pure and applied.[2] At the University of Paris, Poincaré held the titles of Professor of Mathematical Physics and the Calculus of Probabilities, and later Professor of Mathematical Astronomy and Celestial Mechanics. At the Sorbonne, he lectured each year on a different topic, some of which included electricity, light, heat conduction, potential theory, capillarity, elasticity, electromagnetism, hydrodynamics, thermodynamics, optics, celestial mechanics, and probability. In physics, Poincaré is perhaps best known for his work in theoretical dynamics (e.g., the Poincaré recurrence theorem, the Poincaré map, and the discovery of "chaos"), celestial mechanics (e.g., the three-body problem), the theory of space and time (e.g., the principle of relativity), and the dynamics of the electron.[3] Although his pioneering contribution to the quantum theory is lesser known today, it was well known in the early formative years and had a significant impact.

Poincaré was a prolific writer, publishing over 30 books and 500 research papers during his career. His trademark was originality and universality. He was described as "a conqueror, not a colonist."[4] Much of his work involved inventing new fields of mathematics and unifying existing fields. Poincaré wrote several popular books on the nature of science and mathematics.[5,6] These books had a great impact on the public's perception of science, and influenced many scientists and mathematicians.[7] They were read worldwide and translated into six different languages. Under the influence of Poincaré's books, Louis de Broglie changed his major at the University of Paris from history and law to science.[8] In addition to his numerous scientific honors, Poincaré received the highest French honor for literature by being elected to the literary section of the Institute of the French Academy.

Between 30 October 1911 and 3 November 1911, 18 physicists met at the Hotel Metropole in Brussels to participate in the first Solvay Conference. This was the first formal meeting of a distinguished group of invited physicists whose purpose was to elucidate the meaning of the hypothesis of energy quanta and the kinetic theory of matter. Prior to this meeting, the quantum ideas were still obscure and few physicists took them seriously. In a 1910 letter to Walter Nernst (the conference organizer), Max Planck wrote: "Among all those (invited participants) mentioned by you, I believe that other than ourselves, only Einstein, Lorentz, W. Wien, and Larmor will be seriously interested in the topic."[9] Based on this belief, Planck urged Nernst to postpone the conference. In spite of Planck's skepticism, Nernst went ahead with the meeting as planned.

Poincaré arrived at the Solvay Conference a newcomer to the quantum ideas.[10] Little did he know the impact that these new ideas were to have on him. Given that Poincaré was a connoisseur and a certified master of classical physics at the age of 57, one would assume him to be an unlikely convert to the quantum ideas. On the contrary, Poincaré exhibited great enthusiasm and was one of the most active participants in the discussions.[11] Lorentz, in recollections of the conference, writes: "In the discussions, Poincaré displayed all the vivacity and penetration of his mind, and one admired the easiness with which he confidently entered into the more difficult questions of physics, even the ones that were new to him."[12]

In the opening address to the Solvay Congress, Lorentz still called blackbody radiation a "most mysterious phenomenon and a most difficult one to unveil."[13] He urged the participants to "form as clear an idea as possible of the necessity and of the degree of probability of the (quantum) hypothesis."[14] Participants at the Solvay Conference came to the realization that some primitive foundations of the classical theory were in jeopardy. However, they did not agree on what kind of reformulation of classical physics was necessary. During the conference, Poincaré remarked: "The new investigations discussed here do not only seem to question the fundamental principles of mechanics, but also shake violently an assumption (of continuity) which was hitherto completely tied together with the concept of a natural law (of dynamics). Can we still express these laws in the form of differential equations?"[15] Poincaré further says: "It is being asked whether it is necessary to introduce discontinuities into the natural laws, not apparent ones but essential ones."[16]

These remarks by Poincaré summarized a basic, all-encompassing question that permeated the conference: Was the hypothesis of quanta necessary to explain the observed phenomena? Planck and others had only proven that quanta were sufficient: If the hypothesis of quanta is true, then the law of radiation follows. They did not attempt to prove the much more difficult, and stronger, necessary condition: If the law of radiation is true, then the hypothesis of quanta follows. Poincaré was intrigued by the question of the necessity of quanta and regarded it as the vital issue in need of a rigorous proof.

B. Introduction to the memoir

While other physicists were tampering with the sacred classical notion of continuity, Poincaré took it upon himself to prove whether or not the quantum discontinuities were absolutely necessary to explain the observed law of radiation. Immediately after the Solvay Conference, Poincaré returned to Paris and became totally occupied with the quantum problem. On 4 December 1911, one month after the conference, he presented his results to the Academy of Sciences in Paris.[17] The extended version of this work, entitled "Sur la théorie des quanta," appeared in the January 1912 issue of the Journal de Physique.[18] In this memoir, Poincaré proved conclusively that the hypothesis of quanta was both a sufficient and a necessary condition for Planck's law, or any other law of radiation. The quantum discontinuity of nature was thus essential and unavoidable.

The global structure of the memoir is as follows. Poincaré begins with a physical model consisting of a system of Hertzian resonators interacting with a system of atoms. The atoms, by virtue of their collisions with the resonators, mediate the exchange of energy between the resonators. For both physical reasons and mathematical rigor, Poincaré felt that it was necessary to clearly understand how a unique thermodynamic equilibrium state could emerge from an explicit consideration of some mechanism of interaction between the resonators. He felt that the other theoretical studies of blackbody radiation were flawed because they neglected such an interaction and their arguments included too many assumptions related to the existence of equilibrium. He focuses on the partition of energy between the resonators and the atoms in thermal equilibrium. Poincaré's clever choice to study the partition of energy allows him to avoid the usual arguments based on entropy, temperature, and other equilibrium concepts. From his formulation of statistical mechanics, Poincaré is able to relate this macroscopic partition of energy to the microscopic dynamical behavior of a resonator. Having established this basic micro–macro connection, he then determines what kind of constraint must be imposed on the motion (energetics) of the resonators in order to account for the observed partition of energy (radiation law). In three separate proofs, Poincaré discovers that this dynamical constraint is precisely the hypothesis of quanta.

Poincaré's 30-page memoir, which is divided into ten sections, is difficult to read. It is highly mathematical and written in French. The symbolism is quite unlike conventional notation. In trying to summarize the memoir, Sir James Jeans wrote: "Poincaré's paper is of such an abstruse mathematical nature, that it is impossible to do any sort of justice to it in an abstract."[19] Given the high standards of rigor set by Poincaré, together with the formidable task at hand, this complexity is natural. Furthermore, when the memoir first appeared in 1912, the fact that most physicists were unfamiliar with, or misunderstood, the fresh ideas of quanta and statistical mechanics, also contributed to the apparent complexity.

In what follows, I present Poincaré's memoir in a condensed and simpler form. In order to alleviate some of the original complexity, I have reorganized the structure, synthesized the main results, and modernized the arguments, concepts, and notation. Some of Poincaré's eloquent writing style and arguments are inevitably lost in the translation from French to English. Wherever appropriate, I have preserved Poincaré's exact reasoning and notation. Throughout, I have attempted to convey the overall spirit of Poincaré's mode of thinking and to emphasize his originality, physical insight, and impeccable logic.

III. PHYSICAL MODEL AND PARTITION OF ENERGY

A. Poincaré's system of interacting resonators

Poincaré considers a system of resonators, or charged harmonic oscillators. It was well known that such Hertzian resonators, by virtue of their vibration, radiate Hertzian (electromagnetic) waves. Poincaré is extremely careful to address the fundamental issue of thermodynamic equilibrium of this system of resonators. He felt that this crucial issue was seriously overlooked in previous studies.

Poincaré provides a lucid physical description of this concept of equilibrium: "For a (equilibrium) distribution of energy to take place between the resonators of different wavelength whose oscillations are the cause of the radiation, the resonators must be capable of exchanging their energy. Otherwise, the initial distribution would persist indefinitely and, since this initial distribution is arbitrary, there could be no (unique) law of radiation. But, a resonator can give off to the ether and receive from it only light of an exactly determined wavelength... . It could therefore exchange energy only with resonators with which it was in perfect resonance, and the initial distribution would remain unalterable."[20] Poincaré describes the same dilemma with Planck's quantum resonators: "A (Planck) resonator can yield energy to another only in integral multiples of its quantum; the latter can receive energy only in integral multiples of its own quantum. Since the two quanta are generally incommensurate, this is sufficient to exclude the possibility of direct exchange."[21]

In reaction to other proofs of the quantum hypothesis (notably Planck's), Poincaré writes: "First of all, I wondered what was the value of the proposed proofs. I noticed that they were evaluating the probability of the various divisions of energy by simply enumerating them since, thanks to the given hypothesis, they were finite in number, but I could not very well see why they were considered as equally probable. Then, they were introducing the known relations between the temperature, the entropy, and the probability. This assumed the possibility of thermodynamic equilibrium since these relations are proved by assuming this equilibrium to be possible. I know very well that experiment teaches us that this equilibrium is realizable, since it is achieved. But this did not satisfy me; it was necessary to show that this equilibrium is compatible with the stated (quantum) hypothesis and even that it is a necessary consequence of it. I did not exactly have doubts, but I felt the need to see somewhat more clearly, and for this it was necessary to delve a little into the particulars of the mechanism (to establish equilibrium)."[22]

Thus, in order to understand Planck's law of radiation from first principles, Poincaré is compelled to consider the interaction between resonators that is necessary to establish equilibrium. More specifically, Poincaré's grandiose mission is to start from a model which explicitly incorporates the interaction between resonators, and then analyze this model to prove that an equilibrium state exists, and that the hypothesis of quanta is both a sufficient and a necessary condition for the law of radiation.

He proposes two modes of interaction whereby the resonators can exchange energy. First, the stationary resonators can interact with each other indirectly via a medium of freely moving atoms which collide with the resonators. This is a mechanical interaction between resonators mediated by the atoms which can move with a continuum of energies. Poincaré's second suggestion is that the resonators can exchange energy if they are in motion. This electromagnetic interaction exists because radiation reflected from a moving resonator changes its frequency according to the Doppler–Fizeau principle.

In his memoir, Poincaré focuses on the mechanical interaction due to the collisions between resonators and atoms. He emphasizes that this choice is somewhat arbitrary since all modes of interaction should lead to the same equilibrium law of radiation in accord with the second law of thermodynamics. He writes: "...all methods of exchange must lead to the same conditions of statistical equilibrium without which Carnot's principle would be lacking. This is necessary in order to account for experience, but it is still necessary that we be able to give a satisfactory explanation for this surprising concordance, that we not be forced to attribute it to some sort of providential chance. In the older mechanics, this explanation was completely known; it was the universality of Hamilton's equations. Shall we find something analogous here? ... In Mr. Planck's method of exposition, this duality of methods of exchange does not appear, but is merely hidden; and I thought it necessary to call attention to this fact."[23]

Having emphasized the need to understand thermodynamic equilibrium in the new mechanics of the quantum theory, Poincaré considers a system of N resonators interacting with a system of P atoms. All of Poincaré's resonators are one-dimensional, localized, simple harmonic oscillators. In a very clever strategy, Poincaré considers each resonator to have a short period of oscillation, and he models each atom as a resonator with a long period. The long-period resonators have a sufficiently large amplitude so that their motion simulates the uniform rectilinear motion of an atom between collisions. This ingenious idea of modeling the atoms as long-period resonators serves two purposes. First, the simplicity of harmonic motion makes the dynamical analysis tractable. Second, the atoms can be described by classical mechanics since it was well known that low-frequency radiation obeys the classical Rayleigh–Jeans radiation law.[24] Hence, Poincaré's model consists of a system of N short-period (quantum) resonators interacting with a system of P long-period resonators (classical atoms).

B. The weight function

The microscopic dynamical information necessary to understand the macroscopic thermodynamical behavior of the system of resonators is contained in the weight function W. Poincaré defines this function to be the "probability" density of finding the system in each microscopic state. Classically, the microscopic state is specified by the set of coordinates and momenta of each resonator. The collection of microscopic states can be represented as geometric points in state (phase) space. W is then the weight defined on this state space. In dynamical terms, W is proportional to the fraction of time the system spends in each microscopic state during its temporal evolution. Since the equation of motion of a quantum resonator is unknown, Poincaré focuses on the statistics, and not the more detailed dynamics, of the microscopic states.

Poincaré's ultimate goal, to derive an equation for the partition of energy, requires him to know how the weight function W depends on energy. He proves that W for a system of resonators depends only on their energy and can be expressed as a product of the weight functions of each resonator. If $E_1, E_2,...,E_N$ denote the energies of the N resonators, and $U_1, U_2,...,U_P$ denote the energies of the P atoms, then the weight function for the system is

$$W=w(E_1)w(E_2)...w(E_N)v(U_1)v(U_2)...v(U_P). \tag{1}$$

Here, $w(E_i)$ represents the probability weight for a single resonator, labeled i, to oscillate with energy E_i, and $v(U_i)$ is the probability weight for the ith atom to move with an energy U_i. The probability that the system of resonators has a set of energies $E_1,...,E_N$, $U_1,...,U_P$ in the energy volume element $dE_1...dE_N\, dU_1...dU_P$ is proportional to $W\, dE_1...dE_N\, dU_1...dU_P$.

Poincaré performs the transformation from phase space to energy space using his expertise in theoretical dynamics. He analyzes a system of harmonic oscillators interacting via a collisional term to determine how the collision effects the oscillators's statistical/dynamical behavior in state space. In particular, he uses the interpretation of the probability weight as the "last multiplier" of the dynamical equations of motion.[25] Even though the quantum dynamics is not known, Poincaré argues that a last multiplier (equilibrium probability weight) must exist if the concept of equilibrium, or more generally, the second law of thermodynamics, is to make sense. The proof that the weight function factorizes into a product is nontrivial. It is based on an assumption that the collisions are pairwise and occur on a rapid time scale. Nowadays, such a proof that connects the dynamics with the statistics would be considered to be in the difficult domain of ergodic theory. In modern language, Poincaré's result can be understood more simply given the statistical mechanical assumption of assigning the same (unit) weight to each accessible state, together with the quantum mechanical interpretation of the weight function as the density of discrete states.

The weight function W is the fundamental dynamical object of interest in Poincaré's analysis. Together with the conservation of energy, the weight function contains the essential statistical/dynamical information on the motion of the interacting resonators that is necessary to understand their equilibrium behavior. All thermodynamic observables propagate from the knowledge of W via a statistical analysis. In particular, the observed equilibrium value for the energy of a resonator is calculated by averaging over all the microscopic energies using W as the weight.

Poincaré emphasizes that for a classical dynamical system based on Hamilton's equations of motion, the weight function is continuous and assumes a constant value of unity. In other words, equal volumes in phase space are given equal weights. This fact is consistent with Liouville's theorem applied to a system in equilibrium, and is equivalent to Boltzmann's postulate of "equal *a priori* probability" for the classical continuum of microstates accessible to an isolated system.[26] It also leads to the well-known equipartition theorem,[27] whereby the total energy of the system is partitioned equally (on average) among the constituent resonators. This classical theory leads not to Planck's law of radiation, but to the Rayleigh–Jeans law.[24] The Rayleigh–Jeans law does not completely agree with observation, but predicts the well-known "ultraviolet catastrophe" whereby the radiation energy diverges for the high-frequency components.

Poincaré assumes that the system of atoms obeys classical mechanics. Recall that his atoms are modeled as resonators of long period and large amplitude which provide the interaction mechanism, via collisions with the quantum resonators, needed for the system to reach equilibrium. Since low-frequency radiation is well described by classical mechanics (Rayleigh–Jeans law), Poincaré is justified in treating his atoms (low-frequency resonators) classically. Thus, the weight function for the system of classical atoms is uniform and equal to unity. The weight function in Eq. (1) for the system of resonators and atoms then becomes

$$W=w(E_1)w(E_2)...w(E_N). \tag{2}$$

One of Poincaré's primary goals is to find the weight w of a quantum resonator that is both sufficient and necessary to explain the observed law of radiation. This is equivalent to finding the unique energy spectrum of a resonator. Poincaré was soon to discover that such a function is pathologically nonclassical. It must exhibit discontinuities of the quantum kind: w is zero for most energies except for a discrete set of values.

C. The basic partition of energy equation

I now derive a general equation that determines the partition of energy between the quantum resonators and the classical atoms in terms of their microscopic motion. The derivation is based on Poincaré's reasoning, but the final result is expressed in a somewhat different form. In order to relate the microscopic behavior to the macroscopic observable, Poincaré performs a statistical mechanical analysis of his system of interacting resonators and atoms. The weight function contains the crucial dynamical information on the motion of the resonators and the atoms that determines their energy partition.

In focusing on energetics, three energies can be defined:

$$E \equiv E_1+E_2+\cdots+E_N, \tag{3a}$$

$$U \equiv U_1+U_2+\cdots+U_P, \tag{3b}$$

$$H \equiv E+U. \tag{3c}$$

Thus, E is the total energy of the subsystem of N resonators, U is the total energy of the subsystem of P atoms, and H is the total conserved energy of the isolated system of resonators and atoms. The interaction due to the collisions is considered a weak perturbation to the total energy.

The energy of the system of resonators observed in an experiment is the average energy, denoted $\langle E\rangle$. The observed energy of the system of atoms is denoted $\langle U\rangle$. These equilibrium observables are calculated from the weight function W according to the standard definition of average value in mathematical statistics:

$$\langle E\rangle=\frac{\int_H^{H+dH}\cdots\int EW\,dE_1\ldots dE_N\,dU_1\ldots dU_P}{\int_H^{H+dH}\cdots\int W\,dE_1\ldots dE_N\,dU_1\ldots dU_P}, \tag{4a}$$

$$\langle U\rangle=\frac{\int_H^{H+dH}\cdots\int UW\,dE_1\ldots dE_N\,dU_1\ldots dU_P}{\int_H^{H+dH}\cdots\int W\,dE_1\ldots dE_N\,dU_1\ldots dU_P}. \tag{4b}$$

The limits on the integrals denote that the integration is performed over the domain of positive energies consistent with the conservation of energy:

$$H<E_1+\cdots+E_N+U_1+\cdots+U_P<H+dH. \tag{5}$$

In other words, the energies that the resonators and the atoms can move with are constrained so that their total energy $E+U$ is between H and $H+dH$. With the explicit expressions for W, E, and U from Eqs. (2) and (3), the ratio of observable energies in Eq. (4) assumes the form

$$\frac{\langle E\rangle}{\langle U\rangle}=\frac{\int_H^{H+dH}\cdots\int(E_1+\cdots+E_N)w(E_1)\ldots w(E_N)dE_1\ldots dE_N\,dU_1\ldots dU_P}{\int_H^{H+dH}\cdots\int(U_1+\cdots+U_P)w(E_1)\ldots w(E_N)dE_1\ldots dE_N\,dU_1\ldots dU_P}. \tag{6}$$

Poincaré denotes the average energy of one resonator by the symbol y, and the average energy of one atom by the symbol x. Since the total energy is additive over the N resonators, and the P atoms,

$$\langle E\rangle=Ny \quad\text{and}\quad \langle U\rangle=Px. \tag{7}$$

Poincaré's quest is to find the partition of energy in the form of y as a function of x, i.e., $y(x)$.

The integration in Eq. (6) can be simplified by partitioning the integral into a resonator term and an atom term. The partial integration over the atom variables can be performed:

$$\int_U^{U+dU}\cdots\int dU_1\ldots dU_P=\frac{U^{P-1}}{(P-1)!}\,dU. \tag{8}$$

This result, implicit in Poincaré's memoir, is valid for $dU\ll U$, and can now be found in some standard texts on statistical mechanics.[28] The partial integration over the resonator energies cannot be performed explicitly since the unknown weight function depends on these energies. Instead, it is convenient to define a function $g(E,N)$ by

$$\int_E^{E+dE}\cdots\int w(E_1)\ldots w(E_N)dE_1\ldots dE_N\equiv g(E,N)dE. \tag{9}$$

The quantity $g(E,N)$ represents the total weight function for the whole system of N resonators given that the system energy $(E_1+\cdots+E_N)$ is between E and $E+dE$. In the modern language describing a quantum mechanical system, it specifies the countable number of microscopic (quantum) states accessible to the resonator system when it is in the macroscopic state of energy between E and $E+dE$. Planck later called $g(E,N)$ the "thermodynamic probability" to distinguish it from the mathematical probability to which it is proportional. Planck also postulated that this total statistical weight $g(E,N)$, which counts states, is the precise object to use in Boltzmann's expression for the entropy $S=k\ln g(E,N)$, where k is Boltzmann's constant.[29] However, since such a postulate had not been rigorously established, Poincaré does not use the concept of entropy in his memoir and only mentions it in a critical remark. Also recall that Poincaré wanted to avoid any assumptions about equilibrium which are implied in the relations between entropy, temperature, and probability. His partition of energy analysis was specially designed to be entropy-free. This is in sharp contrast to the other pioneering work in this field in which the concept of entropy was central to the arguments.

Finally, the resonator/atom energy ratio in Eq. (6) can be written, using Eqs. (3), (7), (8), and (9), as

$$\frac{Ny}{Px}=\frac{\int_0^H Eg(E,N)(H-E)^{P-1}\,dE}{\int_0^H g(E,N)(H-E)^P\,dE}. \tag{10}$$

This is the basic partition of energy equation that describes how the total energy is partitioned among the interacting resonators and atoms. Poincaré did not write it in this form, but tended to work with separate equations, one for x and one for y, such as those found in Eqs. (4a) and (4b). Our Eq. (10) determines the relationship between the average energy y of a resonator and the average energy x of an atom, given that a total energy H is distributed among N resonators and P atoms. It represents the fundamental connection between the macroscopic partition of energy $y(x)$, and the microscopic motion of the resonator, as embodied in the statistical/dynamical weight function $g(E,N)$.

In the modern language of statistical mechanics, this partition of energy equation can be derived (with less rigor) by starting from the fundamental postulate of "equal *a priori* probability" applied to the isolated system composed of two subsystems (resonators and atoms) in thermal interaction.[30] The probability of finding a particular partition of energy (E and $U=H-E$) between the resonators (R) and the atoms (A) is proportional to the number of discrete microstates accessible to the isolated system which is equal to $g_R(E,N)g_A(H-E,P)$. Summing over all possible energy partitions leads to Eq. (10).

Temperature is one of those superfluous concepts that Poincaré avoids because it is inessential to the analysis. However, he does note the fact that for a classical system, and hence one that obeys the equipartition theorem, the average energy of each constituent particle is proportional to the absolute temperature of the system. Hence, it is implicit in Poincaré's memoir that the average energy x of an atom is equal to the absolute temperature of the system, in appropriate energy units. In other words, the system of atoms serves as a thermometer. The function $y(x)$ then represents the average energy of a resonator as a function of temperature.

Poincaré provides a rigorous proof of the equipartition theorem using his formalism. If the system of resonators obeys classical mechanics, then its weight function W is unity. Similar to the result for classical atoms in Eq. (8), the total statistical weight of a system of classical resonators from Eq. (9) can be written explicitly as

$$g(E,N)dE=\frac{E^{N-1}}{(N-1)!}\,dE. \tag{11}$$

The basic partition of energy [Eq. (10)] can now be integrated exactly to yield the equipartition result[31]

$$y=x. \tag{12}$$

This expresses the classical fact that the total energy of the system is partitioned equally among its constituent degrees of freedom so that the average energy y of each resonator is equal to the average energy x of each atom.

IV. THE THREE PROOFS

A. List of Poincaré's main results

We have so far established Poincaré's basic statistical/dynamical formulation of a system of quantum resonators interacting with a system of classical atoms. This formulation, which culminated in Eq. (10), provides the connection between the macroscopic partition of energy within the whole system and the microscopic dynamics of a single resonator. Armed with this formalism, Poincaré derives many results that are scattered throughout the memoir. To consolidate Poincaré's main results, I identify and extract four of his most important results and rephrase them as the following propositions:

P1 Existence of a unique equilibrium state:
The partition of energy $y(x)$ is independent of the number of resonators (N) and atoms (P) in the thermodynamic limit $N\rightarrow\infty$, $P\rightarrow\infty$, N/P finite.

P2 Sufficient condition for Planck's law:
If the hypothesis of quanta is true, then $y(x)$ is Planck's partition of energy.

P3 Necessary condition for Planck's law:
If $y(x)$ is Planck's partition of energy, then the hypothesis of quanta is true.

P4 Existence of a universal quantum discontinuity:
Given any radiation law, if the total energy of radiation is finite, then there exists a discontinuity in the energy of a resonator.

Prior to Poincaré, physicists focused exclusively on proposition **P2**. The three new propositions proved by Poincaré were vital to the new quantum theory. The proof of **P3** and **P4** settled once and for all the basic controversy regarding the uniqueness of the hypothesis of quanta. Poincaré proved that the known experimental facts can be accounted for by one and only one dynamical hypothesis of an essential quantum discontinuity in the energy.

Poincaré's strong convictions on the fundamental significance of proposition **P1** are described succinctly in his memoir: "Except in very exceptional circumstances, the relation between y and x depends on the integers N and P; but one should consider the case where these integers are very large; even so, it is not at all evident *a priori* that this relationship is independent of the ratio N/P."[32] "If this independence was not true, then thermodynamic equilibrium would not be possible; all the theorems of Boltzmann, which *postulate* the possibility of this equilibrium would not be true; even the notion of entropy would no longer make any sense. To the extent that this independence is not established, it would cast doubt on the reasoning of M. Planck, who relies upon the existence of entropy and the theorems of Boltzmann. That suffices to justify the work that I have presented here."[33] Indeed, our basic partition of energy [Eq. (10)] shows that $y(x)$ depends on N and P. Sir James Jeans, in his objection to Planck's method, also said: "as the ratio (N/P) would vary from one substance to another, there would be no definite law of radiation—the same for all substances—such is demanded by observation."[34]

In the remainder of this paper, these four propositions will be proved in the spirit of Poincaré's original mode of thinking. As discussed before, the arguments and notation will be modified or refined for the sake of simplicity and to make contact with conventional methodology. I divide Poincaré's mathematical analysis into three basic methods which I label as "scaling function proof," "Fourier function proof," and "finite energy proof."

B. Scaling function proof of P1 and P2

Poincaré assumes that the statistical weight of the resonator system has the following functional form for large N:

$$g(E,N)=CF^{N}(E/N)\,\theta(E/N), \tag{13}$$

where F and θ are some functions of the energy per resonator variable E/N, and C is a constant which depends only on N. I call $F(E/N)$ the "scaling function" since it depends on the scaled energy variable E/N. Poincaré proceeds to prove that this scaling conjecture is the most general functional form that will insure that the partition of energy $y(x)$ is independent of N and P, and thus is sufficient for the existence of a unique equilibrium state, independent of the amount of matter. With this scaling representation of $g(E,N)$, the basic partition of energy, Eq. (10), becomes

$$\frac{y}{x}=k\,\frac{\int_0^H[(H-E)^kF(E/N)]^N\theta(E/N)E/(H-E)dE}{\int_0^H[(H-E)^kF(E/N)]^N\theta(E/N)dE}, \tag{14}$$

where k is defined to be the ratio of the number of atoms to the number of resonators:

$$k\equiv P/N. \tag{15}$$

Being an expert in the calculus of probabilities, Poincaré realizes that for large N and nonzero k, the expression raised to the power N in both integrands of Eq. (14) has a pronounced maximum which dominates the integrals. This maximum determines the most probable value of E, which Poincaré identifies with the equilibrium state. If the integrals in Eq. (14) are approximated by their maximum term, then the most probable E is identical to the average E. Thus, the average energy of a resonator y is determined by the equilibrium condition

$$(H-Ny)^kF(y)=\text{maximum}. \tag{16}$$

Upon setting the (logarithmic) derivative equal to zero to find the maximum, Eq. (16) is equivalent to

$$\frac{Nk}{H-Ny}-\frac{F'(y)}{F(y)}=0, \tag{17}$$

where $F'(y)$ denotes the derivative of $F(y)$. Using conservation of energy, $H=Ny+Px$, and the definition $k\equiv P/N$, Eq. (17) becomes

$$\frac{F(y)}{F'(y)}=x. \tag{18}$$

This is Poincaré's basic equation that implicitly determines the macroscopic partition of energy $y(x)$ in terms of the scaling function $F(y)$ representation of the microscopic statistical weight $g(E,N)$. Note that all dependence on k has vanished. Thus not only has Poincaré found a simple relation between $y(x)$ and $g(E,N)$, but he has also found the general functional form of $g(E,N)$ that is sufficient for the concept of equilibrium to make sense in the thermodynamic limit.

Poincaré devotes a section of his memoir to using his scaling function formulation to derive Planck's law from the hypothesis of quanta. According to the quantum hypothesis, the energy of a resonator is a multiple of the quantum of energy ϵ, which is a constant.[35] Thus, the weight function $w(E_1)$ for one resonator of energy E_1 is zero for all values of E_1 that are not a multiple of ϵ. Poincaré further concludes that if E_1 is a multiple of ϵ, then $w(E_1)$ diverges to infinity in such a way that the integral

$$\int_a^b w(E_1)dE_1 \tag{19}$$

is equal to the number of multiples of ϵ between a and b. This is Poincaré's translation of Planck's quantum hypothesis into the language of his weight function.[36]

Poincaré uses this quantum weight function $w(E_1)$ for one resonator to calculate the statistical weight $g(E,N)$ for the total system of N resonators. He notes that for this quantum weight function, the integral expression defining $g(E,N)$ in Eq. (9) turns into a finite sum which merely counts the number of ways N integers (positive or zero) can sum to the value E/ϵ. With this interpretation, Poincaré immediately writes the formula

$$g(E,N)=\frac{(E/\epsilon+N-1)!}{(E/\epsilon)!(N-1)!}. \tag{20}$$

In modern quantum words, this is the classic combinatoric problem of distributing E/ϵ indistinguishable quanta among N distinguishable resonators. Thus Poincaré's entropy-free analysis recovers Planck's "entropic" result[37] with much more rigor. The thermodynamic limiting form of $g(E,N)$ for large N and large E/ϵ can be found upon approximating the factorials in Eq. (20) using Stirling's formula.[38] One finds that $g(E,N)$ has precisely the scaling form necessary for equilibrium, as postulated in Eq. (13), where the scaling function assumes the specific form[39]

$$F(y)=\left(1+\frac{\epsilon}{y}\right)^{y/\epsilon}\left(1+\frac{y}{\epsilon}\right). \tag{21}$$

Having calculated this scaling function based on the quantum hypothesis, the equilibrium partition of energy is generated from $F(y)$ by taking its logarithmic derivative according to the general rule in Eq. (18). The result is

$$y=\frac{\epsilon}{e^{\epsilon/x}-1}. \tag{22}$$

This is Planck's energy partition formula. Note that this partition of energy is entirely different from the classical equipartition result $y=x$. The classical result can be recovered from Eq. (22) in the continuum limit $\epsilon\to 0$. Thus, based on the quantum hypothesis and the scaling assumption, Poincaré's statistical/dynamical theory of interacting resonators and atoms leads to Planck's law.

C. Fourier function proof of P1, P2, and P3

In his memoir, Poincaré calls this section his "second method." He is aware of the limitations of his first (scaling function) method, and still wants to know: "(1) if the law of partition is independent of the ratio N/P for any function w; (2) if the hypothesis (of quanta) is the only one that leads to the law of Planck."[40] Poincaré's quest is to develop a more general formalism that will allow him to answer these questions with complete rigor. He writes: "To answer these two questions, I will use another mode of calculation, based on the integral of Fourier."[40] Poincaré introduces a function Φ defined by

$$\Phi(\alpha)\equiv\int_0^\infty w(E_1)e^{-\alpha E_1}\,dE_1, \tag{23}$$

where α is a complex variable whose real part is positive, and $w(E_1)$ is the weight function of one resonator of energy E_1. This equation can be inverted to give[41]

$$w(E_1)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(\alpha)e^{\alpha E_1}\,d\alpha, \tag{24}$$

where i is $\sqrt{-1}$, and the integration path in the complex α plane is the vertical line Re $\alpha=a$ such that a is any real positive number. Note that Poincaré refers to this method as due to Fourier, because the inversion of Eq. (23), which is a Laplace transform in modern language,[41] is based on Fourier's integral. I call Poincaré's Φ function the "Fourier function."

Poincaré sets out to find the relation between the equilibrium partition of energy $y(x)$ and his Fourier function $\Phi(\alpha)$. If Eq. (23) is multiplied by itself N times, then

$$\Phi^N(\alpha)=\int_0^\infty\cdots\int_0^\infty w(E_1)\ldots w(E_N)\times e^{-\alpha(E_1+\cdots+E_N)}dE_1\ldots dE_N. \tag{25}$$

By virtue of the definition of $g(E,N)$ in Eq. (9) and the definition of E in Eq. (3a), this can be simplified to

$$\Phi^N(\alpha)=\int_0^\infty g(E,N)e^{-\alpha E}\,dE. \tag{26}$$

Thus the Fourier function for the whole system of N resonators [right-hand side of Eq. (26)] is simply the product of the Fourier functions of each resonator (left-hand side). The inversion of Eq. (26) yields the statistical weight for the system of N resonators of total energy E:

$$g(E,N)=\frac{1}{2\pi i}\int_C\Phi^N(\alpha)e^{\alpha E}\,d\alpha, \tag{27}$$

where C denotes the vertical-line contour described above in conjunction with Eq. (24). Using this Fourier representation of the statistical weight, the basic partition of energy Eq. (10) becomes

$$\frac{y}{x}=k\,\frac{\int_0^H\int_C[(H-E)^k e^{\alpha E/N}\Phi(\alpha)]^N E/(H-E)d\alpha\,dE}{\int_0^H\int_C[(H-E)^k e^{\alpha E/N}\Phi(\alpha)]^N d\alpha\,dE}, \tag{28}$$

where k is the ratio P/N. It is instructive to compare Eq. (28) to the analogous equation (14) based on the scaling function method.

As before, Poincaré identifies the equilibrium state with the most probable state. For large N, the integrals in Eq. (28) are dominated by the pronounced maximum of the same term in their integrands that is raised to the power N. Thus, the equilibrium condition is

$$(H-Ny)^k e^{\alpha y}\Phi(\alpha)=\max. \tag{29}$$

Upon setting the (logarithmic) differential equal to zero, one finds that the most probable y and α are determined by

$$\left(\alpha-\frac{Nk}{H-Ny}\right)dy+\left(\frac{\Phi'(\alpha)}{\Phi(\alpha)}+y\right)d\alpha=0. \tag{30}$$

Using the conservation of energy, $H=Ny+Px$, and the definition $k\equiv P/N$, it follows that the equilibrium condition that determines the resonator energy y and the atom energy x is

$$y=-\frac{\Phi'(\alpha)}{\Phi(\alpha)},\quad x=\frac{1}{\alpha}. \tag{31}$$

These are Poincaré's basic equations that implicitly determine (by eliminating α) the macroscopic partition of energy $y(x)$ in terms of the Fourier function $\Phi(\alpha)$ representation of the microscopic weight function $w(E_1)$. Note that the equilibrium partition of energy is independent of k. Poincaré's physical model and mathematical machinery provide clear and explicit insight into this disappearance of k. This justifies the assumption of equilibrium made by Planck and others. It represents Poincaré's rigorous proof of **P1**.

Poincaré's Fourier methodology is a general and powerful statistical mechanical formalism. Unlike the scaling function method, it is exact, since it involves no assumptions about the functional form of $g(E,N)$. It is computationally simple, since the Fourier function $\Phi(\alpha)$ is determined by one resonator, unlike $g(E,N)$ which depends on the whole system of N resonators constrained by energy conservation. The crucial mathematical feature of Poincaré's Fourier method, that is tailor-made for his purpose, is the invertibility of the Fourier integral, i.e., the uniqueness of the Fourier transform pair of functions (Φ and w). Given one of these functions, the other is uniquely determined.

The classical equipartition theorem can be proved with elegant simplicity using Poincaré's new method. If the weight function $w(E_1)=1$, then the Fourier function is

$$\Phi(\alpha)=\int_0^\infty e^{-\alpha E}\,dE=\frac{1}{\alpha}, \tag{32}$$

and hence the partition of energy from Eq. (31) is simply

$$y=x. \tag{33}$$

Thus, the total energy of a system of classical resonators in equilibrium is partitioned equally (on average) among the constituent resonators.

Of course, Poincaré's primary application of his new theory is to understand the connection between the observed (nonclassical) partition of energy and the hypothesis of quanta. If the quantum hypothesis is true, then the discontinuous weight function defined in Eq. (19) transforms the Fourier integral of Eq. (23) into the geometric series

$$\Phi(\alpha)=\sum_{n=0}^{\infty} e^{-n\alpha\epsilon}, \tag{34}$$

which sums exactly to

$$\Phi(\alpha)=\frac{1}{1-e^{-\alpha\epsilon}}. \tag{35}$$

The partition of energy can easily be generated from this explicit form of the Fourier function according to the general equation (31). The result is

$$y=\frac{\epsilon}{e^{\epsilon/x}-1}. \tag{36}$$

This is Planck's formula. Thus the hypothesis of quanta is a sufficient condition for Planck's law to be true. Although Poincaré was not the first to prove this sufficiency, his proof was based on first principles, and it was one of the simplest and the most convincing. It was distinguished from the other proofs by its mathematical rigor, its explicit physical consideration of the interaction between resonators, and its focus on the partition of energy. Furthermore, it made no assumptions on equilibrium, counting states, or probability, and did not rely on the concepts of entropy or temperature.

Poincaré identifies his Fourier parameter α with $1/T$ only in a passing remark. This is due to the fact that Poincaré's classical resonators (atoms) have an average energy $x=1/\alpha$ according to Eq. (31). However, as we have stressed before, Poincaré focuses on the partition of energy and avoids any extraneous concepts that would distract from the essential logic.

Poincaré was most interested in proving what no one had proved before, namely, that the hypothesis of quanta was a necessary condition for Planck's law. Given the power and versatility of his Fourier formalism, Poincaré proves the necessity of quanta in one paragraph. He writes: "When the law that connects y and x is given, the logarithmic derivative Φ'/Φ is also known; it is therefore the same as the function Φ to within a constant factor, and as a consequence [by Eq. (24)], the same w. *The hypothesis of quanta is therefore the only one that leads to the law of Planck.*"[42] In other words, Poincaré's Fourier formalism allows him to simply invert the sufficiency proof ($w\rightarrow\Phi\rightarrow y$) to arrive at the necessity proof ($y\rightarrow\Phi\rightarrow w$).

Poincaré's wise decision to work in Fourier space was fruitful for several reasons. The exponential function $\exp(-\alpha E)$ in the Fourier integral has wonderful algebraic properties. It is responsible for the partitioning of the Fourier function for the whole system of resonators into a product over the Fourier functions of each constituent resonator as shown in Eq. (26). In modern statistical mechanical language, this exponential function is called the Boltzmann factor.[43] Poincaré is the first to simply introduce it as the kernel of an integral transform. The unique connection between the transform pair of functions (Φ and w) is the powerful trademark of the Fourier theory that allows Poincaré to prove both the sufficiency and the necessity of the quantum hypothesis. Poincaré is also the first to invent this general statistical mechanical formalism that allows one to derive the microscopic dynamical behavior, given the macroscopic thermodynamical behavior. Since this is the inverse of the usual program that derives macroscopic output from microscopic input, I call Poincaré's formalism an "inverse statistical mechanics."

D. Finite energy proof of P4 (universality proof)

The last section of Poincaré's memoir is devoted to a general proof that a quantum discontinuity is necessary to explain the blackbody phenomena, *independent* of the specific form of the radiation law. In particular, Poincaré proves that for any observed radiation law, the mere physical fact that the total radiation energy must be finite requires that there exist a discontinuity in the dynamical motion (energetics) of a resonator. The motivation for this proof is clearly described by Poincaré: "An experimental law is never more than an approximation; could we then imagine laws whose difference from Planck's law would be within the errors of observation and which would lead to a continuous function $w(E_1)$?"[42] In this universality proof, which is based on his Fourier formalism, Poincaré does not make any reference to Planck's law or any other specific law of radiation. However, he does assume the validity of a general thermodynamical result (due to Wien) and a general electrodynamical result (due to Planck).

Poincaré begins with the well-known relation,[44] established by Planck, between the spectral energy density u_λ of radiation of wavelength λ and the average energy y of a resonator of the same frequency, that are in thermal equilibrium with each other:

$$u_\lambda = \frac{K}{\lambda^4} y. \tag{37}$$

The quantity $u_\lambda\, d\lambda$ is the energy of radiation per unit volume contained in the wavelength interval between λ and $\lambda + d\lambda$, and K is a constant. Using Eq. (31), this energy density can be generated from the Fourier function Φ according to

$$u_\lambda = \frac{K}{\lambda^4} x^2 \frac{d \log \Phi}{dx}. \tag{38}$$

Poincaré next analyzes the general functional form of Φ as it depends on x and λ. He utilizes the well-known displacement law of Wien[45] which states that the general functional form of the energy density is

$$u_\lambda = \frac{1}{\lambda^5} F(\lambda T), \tag{39}$$

where F is some unspecified function of the product of wavelength λ and absolute temperature T. Identifying the average energy x of an atom (classical long-wavelength resonator) with the temperature T, Eqs. (38) and (39) yield

$$d \log \Phi = \frac{F(\lambda x)}{K \lambda x} \frac{dx}{x}. \tag{40}$$

Note that the right-hand side of this equation is invariant upon changing x to μx and λ to λ/μ, where μ is a constant. This means that Φ must be a function of a single variable which is the product λx, and thus[46]

$$x \frac{d \log \Phi}{dx} = \lambda \frac{d \log \Phi}{d\lambda}. \tag{41}$$

The spectral energy density in Eq. (38) becomes[47]

$$u_\lambda = K \frac{x}{\lambda^3} \frac{d \log \Phi}{d\lambda}. \tag{42}$$

Hence, the total radiation energy is determined by the Fourier function Φ according to

$$\int_0^\infty u_\lambda\, d\lambda = Kx \int \frac{d \log \Phi}{\lambda^3}. \tag{43}$$

This equation could predict an absurd nonphysical result of an infinite radiation energy whenever the integral on the right-hand side diverges due to the behavior of the integrand near $\lambda=0$. The Φ function near $\lambda=0$ holds the crucial information about the overall behavior. This short-wavelength regime corresponds to the radiation emitted by Poincaré's short-period (quantum) resonators. It is precisely these nonclassical resonators that he has analyzed so carefully, based on their interaction with the long-period (classical) resonators.

Poincaré therefore focuses his attention on the analytic behavior of the Φ function in the pathological domain near $\lambda=0$. Since he does not assume any specific form for the radiation law, this Fourier function is also unknown. However, using an ingenious argument, Poincaré establishes an upper bound on Φ at $\lambda=0$. He first partitions the definition of Φ in Eq. (23) into two integration domains:

$$\Phi(\alpha) = \int_0^{E_0} w(E) e^{-\alpha E}\, dE + \int_{E_0}^\infty w(E) e^{-\alpha E}\, dE, \tag{44}$$

where E_0 is any value, and for convenience I have denoted the energy of one resonator E_1 by E. Since α is positive, the first integral, denoted I_1, satisfies

$$I_1 < \int_0^{E_0} w(E) dE. \tag{45}$$

In the second integral, denoted I_2, let $\alpha = \alpha_0 + \alpha'$ where α_0 is a constant, and then note that

$$I_2 < e^{-\alpha' E_0} \int_{E_0}^\infty w(E) e^{-\alpha_0 E}\, dE < e^{-\alpha' E_0} \Phi(\alpha_0). \tag{46}$$

Together, these inequalities imply that

$$\Phi(\alpha) < \int_0^{E_0} w(E) dE + e^{-\alpha' E_0} \Phi(\alpha_0). \tag{47}$$

The behavior of Φ at $\alpha = \infty$ (or equivalently $\alpha' = \infty$) is identical to its behavior at $\lambda=0$ since Φ is a function of the product $\lambda x = \lambda/\alpha$. Thus the ultraviolet Fourier function is bounded according to

$$\Phi(\lambda=0) < \int_0^{E_0} w(E) dE, \tag{48}$$

where E_0 can assume any value.

This bound on $\Phi(\lambda=0)$ provides the crucial information that determines what kind of constraint must be imposed on the motion of a resonator to insure that the total radiation energy is finite. If $w(E)$ is continuous, then $\Phi(\lambda=0)=0$, since the Fourier integral in Eq. (23) tends to zero as α tends to infinity. This also follows from Eq. (48) by letting E_0, which is arbitrary, tend to zero. But if $\Phi=0$, then $\log \Phi$ diverges and hence the total energy from Eq. (43) diverges even more rapidly. More specifically, in the ultraviolet regime (near $\lambda=0$), the energy integral diverges like $\log \Phi(\lambda)/\lambda^3$. This is the explicit realization of the "ultraviolet catastrophe." Thus, if $w(E)$ is continuous, then the total energy of radiation is infinite. In order for the total energy to be finite, Φ cannot be zero for $\lambda=0$. If $\Phi(\lambda=0) \neq 0$, then Poincaré's inequality in Eq. (48) implies that $w(E)$ cannot be continuous. In his memoir, Poincaré's logical conclusion is that "if $\Phi(\lambda)$ remains finite for $\lambda=0$, *and that will happen whenever the law of radiation is such that the total radiation energy is finite*, then the integral $\int_0^{E_0} w(E)dE$ must remain finite when E_0 tends to zero; that is to say that the function $w(E)$ must exhibit for $E=0$ the same kind of discontinuity as in the hypothesis of Planck, and that excludes the possibility of representing the phenomena by differential equations."[48]

Poincaré has thus proved that for any radiation law, the physical fact that the total energy of radiation is finite requires that the motion of the resonator, as described by $w(E)$, must exhibit a discontinuity of the quantum kind. Although his proof was quite general, Poincaré does remark that there remains some doubt about its complete rigor since it utilized the result of Planck in Eq. (37) that is based on classical electrodynamics.

V. IMPACT OF THE MEMOIR

Poincaré's proof of the necessity of quanta played a significant role in the development of quantum theory, primarily because of its strong influence on the scientific community. It convinced many skeptical scientists, clinging to the cherished classical ideas, to finally believe in the quantum ideas. Poincaré's mathematical machinery, clever reasoning, and concrete physical model of interacting resonators, combined to produce a compelling proof. The international reputation and unquestioned authority of this renowned mathematical physicist also helped to influence the skeptics. Unlike the work of others during these early years of the quantum theory, most notably Einstein and Ehrenfest, Poincaré's paper did not go unnoticed and was an often cited memoir.[49] de Broglie wrote: "Historically, the conclusions of Poincaré have played a large role in the development of the quantum theory...to show the existence of a quantum discontinuity."[50]

Prior to 1911, the domain of quantum theory was almost exclusively in Germany. Poincaré's publication helped to propagate the quantum ideas and gain their acceptance beyond Germany's borders. Indeed, after the January 1912 publication date of the memoir, the French publications on quantum topics increased and remained continuous.[51]

Poincaré's proof had the most dramatic impact in Great Britain.[52] The British scientists were some of the strongest opponents of the quantum theory. The most influential among these opponents were Sir Jeans, Sir Thomson, and Lord Rayleigh. On September 12, 1913, the British Association held a meeting in Birmingham in order to discuss the quantum ideas. It was the British version of the Solvay Conference in Brussels. In Brussels, Jeans presented strong arguments against the quantum theory. In Birmingham, Jeans publicly announced his complete acceptance of the quantum theory. Poincaré's proof was clearly the reason for his conversion. "Mr. Jeans regarded the work of Poincaré as conclusive."[53] In the opening address, Jeans announced that because of Poincaré, he felt "logically compelled to accept the quantum hypothesis in its entirety."[54] Subsequently, Jeans became the British authority and spokesperson on the quantum theory. In 1914, he published the book entitled *Report on Radiation and the Quantum Theory*,[55] in which Poincaré's influence was clearly evident. This 90-page book was the first comprehensive textbook on the subject of quantum mechanics. For ten years, it served as the standard English language source on the subject. Thus Poincaré's direct influence on Jeans resulted in the quantum education of a vast number of English-speaking scientists during the formative years of the quantum theory.

In addition to its role in converting quantum skeptics to quantum believers, Poincaré's memoir had substantial repercussions on the subject of statistical mechanics. In particular, Poincaré's memoir helped make the statistical mechanical ideas familiar to physicists in the early decade of the 20th Century.[56] Boltzmann's original papers and Gibbs' pioneering book[57] remained rather obscure because of the difficulties inherent to the subject. Furthermore, Gibbs' book was considered difficult to understand by scientists, including Einstein[58] and Poincaré.[59] Both Einstein and Poincaré were well aware of the fundamental importance of statistical mechanics and devoted much effort to reformulating it more clearly. A wonderfully refreshing, personalized version of statistical mechanics, based on the calculus of probabilities and the theory of Fourier, permeates the memoir. Poincaré's version comes close to the modern-day presentation of the subject.[60] In this modern language, Poincaré's g function is the multiplicity function and his Fourier function Φ is the partition function. The Fourier variable α is the inverse temperature. His scaling function method corresponds to the microcanonical formulation. His Fourier function method corresponds to a unique hybrid formulation that convolutes the microcanonical (g for the classical atoms) and the canonical (Φ for the quantum resonators) methods. Poincaré's technique of generating observables (average values) from logarithmic derivatives of g or Φ is now a standard trademark of modern statistical mechanical methodology.

Stimulated by Poincaré's statistical mechanical formalism, Planck was the first to call Poincaré's Fourier function Φ the "Zustandsintegral" (integral-over-states), or the "Zustandsumme" (sum-over-states), in his 1921 paper entitled "Henri Poincaré und die Quantentheorie."[61] Planck, like Poincaré, realized that this function was the fundamental object of the theory because it contained the complete microscopic information necessary to generate all equilibrium thermodynamic observables. In 1924, Planck introduced the symbol Z for this function.[62] This notation and the central role of Z in the subject of statistical mechanics have remained ever since.

In 1921, Fowler refined Poincaré's mathematical treatment of quantum discontinuities using a generalized Fourier integral, known as the Stieltjes' integral.[63] Strictly speaking, Poincaré's manipulation of a nonanalytic weight function within the framework of Fourier theory was not legitimate, since the rigorous mathematics of such discontinuous (generalized) functions was not developed until much later. One can only speculate that Poincaré was well aware of this pathology and was confident in his analysis without commenting on the intricacies. In 1926, Dirac, who was at the time Fowler's Ph.D student at Cambridge, introduced the delta function for use in quantum physics. In mathematics, this function was not considered a well-defined mathematical entity until 1945.[64] With the Dirac delta-function representation of quantum discontinuities, Poincaré's Fourier method was completely justified.

Poincaré introduced complex variables and Fourier integral methods into the subject of statistical mechanics, thereby creating an "inverse statistical mechanical theory." This theory allows one to derive the microscopic behavior of the constituent particles of matter from the observed macroscopic phenomena. This is the inverse of the conventional path of logic in statistical mechanics whereby the macroscopic behavior is derived from the microscopic behavior.

In 1922, Fowler, together with Darwin (the grandson of evolutionist Charles Darwin), developed a powerful statistical mechanical formalism that also used complex variables and contour integrals.[65] Like Poincaré, they focused on the partition of energy, but their formalism was more general. In particular, their complex variable technique was designed to calculate average values, not most probable values, and could be applied to any atomic system that obeyed the laws of quantum dynamics. They were the first to introduce the terminology "partition function" for the fundamental object of their theory that was equivalent to Poincaré's Fourier function and to Planck's Zustandsumme, or to Gibbs' phase integral for classical systems. In 1929, Fowler published his classic *Statistical Mechanics*[66] which was the first comprehensive textbook on statistical mechanics that incorporated the new quantum mechanics as the mechanical foundation. A section of the book is devoted to Poincaré's inverse statistical mechanical theory.[67]

VI. CONCLUSIONS

Thanks to Poincaré, quanta were inescapable. Continuity, which was the trademark of classical physics, must be replaced by one of discontinuity in the new quantum physics. Poincaré's proof was unique among the other pioneering studies of the hypothesis of quanta. It was the first to provide a concrete physical understanding of the existence of a unique thermodynamic equilibrium state of the system of Hertzian resonators from an explicit analysis of an interaction mechanism (atomic collisions). It was free of temperature, entropy, and the second law, since Poincaré's partition of energy analysis allowed him to bypass these concepts. It was the first proof of the necessity of quanta, demonstrating that the hypothesis of quanta was the only hypothesis that could account for Planck's law of radiation. The finite energy proof demonstrated that a quantum discontinuity was essential to natural dynamical law. It was the most universal proof, making no reference to any particular law of radiation. To accomplish these necessity proofs, Poincaré invented an "inverse statistical mechanics." This general formalism allows one to operate the traditional statistical mechanical machinery in reverse, so that the microscopic dynamics can be derived from the macroscopic thermodynamics. In spite of the mathematical nature of the memoir, Poincaré's attention to physical details (interaction mechanism), flawless logic (mathematical rigor), and keen focus on the essentials (avoiding temperature and entropy) endow the proof with a clarity and a power that render it absolutely convincing. Indeed, the memoir played a pivotal role in the development of both quantum mechanics and statistical mechanics.

Poincaré's memoir is full of fresh physical and mathematical ideas that are as rare and innovative today as they were in 1912. Most notable of these are the physical model of short-period resonators interacting with long-period resonators, the focus on the partition of energy, the simple proofs of the equipartition theorem, the use of Fourier theory/complex analysis in statistical mechanics, the inverse statistical mechanical theory, and finally, the scaling, necessity, and universality proofs themselves. This unique blend of physical and mathematical ideas, when incorporated into a course on quantum mechanics, statistical mechanics, or mathematical physics, can augment and enhance the standard presentations, while providing a substantial boost in the amount of convincing rigor.

Poincaré's celebrated proof was his first excursion into the realm of quantum theory following a life-long devotion to classical physics. It was also his last memoir on mathematical physics. Poincaré died on 17 July 1912, just six months after the publication of his proof. This last memoir, which marked the end of Poincaré's life, also marked the end of the classical era of continuity in natural law.

In the introduction to his memoir, Poincaré writes: "It is hardly necessary to remark how much this concept (of quanta) differs from what we imagine to be true up to this point; physical phenomena would cease to obey the laws expressed as differential equations and this would undoubtedly be the greatest and the most profound revolution that natural philosophy has undergone since Newton."[68] Indeed, Poincaré believed that the notion of continuity in the mathematical description of nature was one of the most fundamental hypotheses of mathematical physics. He wrote that the hypothesis of continuity is one of the "last that ought to be abandoned."[69] He also writes "a belief (in continuity) would be difficult to justify by apodeictic reasoning, but without which all science would be impossible."[70] In spite of these strong convictions, Poincaré did in fact abandon this precious classical idea to which he had devoted himself most of his life. Planck, in his publication "Henri Poincaré und die quantentheorie,"[71] wrote "The old man will be inclined to ignore the hypothesis, the enthusiast will welcome it uncritically, the skeptic will seek grounds to reject it, the productive man will test it, and if possible, fructify it. Poincaré, in the profound paper which he dedicated to the quantum theory, proved himself youthful, critical, and productive."

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[15]J. Mehra and H. Rechenberg, Ref. 8, p. 135.
[16]H. Poincaré, Ref. 5, p. 76.
[17]H. Poincaré, "Sur la théorie des quanta," C. R. Acad. Sci. (Paris) **153**, 1103–1108 (1911).
[18]H. Poincaré, "Sur la théorie des quanta," J. Phys. (Paris) **2**, 5–34 (1912).
[19]J. Jeans, *Report on Radiation and the Quantum Theory*, 2nd ed. (Fleetway, London, 1924), p. 24.
[20]H. Poincaré, Ref. 5, pp. 96–97.
[21]H. Poincaré, Ref. 5, p. 82.
[22]H. Poincaré, Ref. 5, p. 96.
[23]H. Poincaré, Ref. 5, pp. 81–82.
[24]M. Jammer, Ref. 7, pp. 12–13.
[25]R. H. Fowler, *Statistical Mechanics*, 2nd. ed. (Cambridge University, Cambridge, 1936), p. 14.
[26]K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, New York, 1987), pp. 127–130; R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972), pp. 32–36.
[27]M. Jammer, Ref. 7, pp. 9–11.
[28]This definite integral is a special case of Dirichlet's integral and is discussed in R. C. Tolman, *The Principles of Statistical Mechanics* (Dover, New York, 1979), pp. 491–492. A geometric interpretation of the integral as the hypervolume bounded between diagonal hyperplanes is discussed in H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (Wiley, New York, 1985), pp. 343–348.
[29]M. Planck, *The Theory of Heat Radiation* (Dover, New York, 1991), pp. 118–126.
[30]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw–Hill, New York, 1965), Chaps. 2 and 3; C. Kittel and H. Kroemer, *Thermal Physics*, 2nd ed. (Freeman, San Francisco, 1980), Chap. 2; K. Stowe, *Introduction to Statistical Mechanics and Thermodynamics* (Wiley, New York, 1984), Chaps. 7 and 8.
[31]Note that for this equipartition, $y=H/(N+P)$ and $x=H/(N+P)$. This can be proved more formally from the explicit evaluation of the integrals in Eq. (4) with $W=1$.
[32]H. Poincaré, Ref. 17, p. 1105.
[33]H. Poincaré, Ref. 18, p. 18.
[34]J. Jeans, Ref. 19, p. 22.
[35]Poincaré does not assume that the quantum of energy ϵ is proportional to the frequency ν of the resonator, as it is in Planck's hypothesis where $\epsilon=h\nu$, with h being Planck's constant.
[36]If Poincaré had access to the Dirac delta function, then he could have defined his weight function more concisely by $w(E_1)=\Sigma_{n=0}^{\infty}\delta(E_1-n\epsilon)$.
[37]M. Planck, Ref. 1, p. 41; M. Jammer, Ref. 7, p. 15.
[38]Stirling's formula is $n!=n^n e^{-n}(2\pi n)^{1/2}$ for $n\gg 1$. See F. Reif, Ref. 29, Appendix A6.
[39]The constant and the other scaling function in Eq. (13) can be shown to be $C=1/(2\pi N)^{1/2}$ and $\theta(y)=\epsilon/y(y+\epsilon)^{1/2}$.
[40]H. Poincaré, Ref. 18, p. 23.
[41]M. L. Boas, *Mathematical Methods in the Physical Sciences*, 2nd ed. (Wiley, New York, 1983), pp. 662–663.
[42]H. Poincaré, Ref. 18, p. 27.
[43]C. Kittel and H. Kroemer, Ref. 30, p. 58.
[44]M. Jammer, Ref. 7, p. 9 and Appendix A.
[45]M. Jammer, Ref. 7, p. 6.
[46]For the special case of Planck's radiation law, Poincaré's general argument that Φ depends on λx is equivalent to showing that $\epsilon=h\nu$. Poincaré does not mention this.
[47]The analogous equation in Poincaré's memoir (Ref. 18, p. 29) contains a typographical error.
[48]H. Poincaré, Ref. 17, pp. 1107–1108.
[49]R. McCormmach, Ref. 10, p. 51; for a summary of Ehrenfest's work, see M. J. Klein, *Paul Ehrenfest: The Making of a Theoretical Physicist* (North-Holland, Amsterdam, 1970), pp. 245–253. In particular, a theorem on the necessity of quanta for Planck's radiation law was stated without proof by Ehrenfest in October of 1911. In the same paper, Ehrenfest did provide a detailed proof that a finite total energy of radiation implies a discontinuity in the energy spectrum of a resonator at zero energy. The proof was quite different than Poincaré's. It was also in this paper that Ehrenfest introduced the phrase "ultraviolet catastrophe." It is unfortunate that Ehrenfest's pioneering work did not attract much attention.
[50]L. de Broglie, in the preface to *Oeuvres de Henri Poincaré: Physique Mathématique* (Gauthier-Villars, Paris, 1954), Vol. 9, pp. xi–xii.
[51]T. S. Kuhn, *Blackbody Theory and the Quantum Discontinuity 1894–1912* (University of Chicago, Chicago, 1978), pp. 210 and 231.
[52]R. McCormmach, Ref. 10, p. 53.
[53]"Physics at the British Association," Nature **92**, 304–308 (1913).
[54]J. Mehra and H. Rechenberg, Ref. 8, p. 152. Although Jeans was perceived as a quantum skeptic in public, it can be argued that he was partially convinced of the quantum theory before Poincaré's memoir, and that Poincaré's proof finally convinced Jeans rather than completely converted him (R. McCormmach, private communication).
[55]J. Jeans, *Report on Radiation and the Quantum Theory* (The Electrician Printing and Publishing, London, 1914).
[56]S. G. Brush, *Statistical Physics and the Atomic Theory of Matter, From Boyle and Newton to Landau and Onsager* (Princeton University, Princeton, NJ, 1983), p. 78.
[57]J. W. Gibbs, *Elementary Principles in Statistical Mechanics* (Ox Bow, Woodbridge, CT, 1981).
[58]A. Einstein, book review of "Les théories statistique en thermodynamique" by H. A. Lorentz, Naturwiss. **4**, 480–481 (1916).
[59]H. Poincaré, Ref. 6, p. 304.
[60]For modern presentations, see the textbooks in Ref. 30.
[61]M. Planck, "Henri Poincaré und die quantentheorie," Acta Math. **38**, 387–397 (1921).
[62]S. G. Brush, Ref. 56, p. 78; Max Planck, *Theory of Heat* (Macmillan, London, 1932), pp. 244–245.
[63]R. H. Fowler, "A simple extension of Fourier's integral theorem and some physical applications, in particular to the theory of quanta," Proc. R. Soc. London Ser. A **99**, 462–471 (1921).
[64]M. Jammer, Ref. 7, p. 328.
[65]R. H. Fowler and C. G. Darwin, "On the partition of energy," Philos. Mag. **44**(6), 450–479 (1922).
[66]R. H. Fowler, *Statistical Mechanics: The Theory of the Properties of Matter in Equilibrium* (Cambridge University, Cambridge, 1929).
[67]R. H. Fowler, Ref. 66, Sec. 6.7.
[68]H. Poincaré, Ref. 18, p. 5.
[69]H. Poincaré, Ref. 6, p. 135.
[70]H. Poincaré, Ref. 6, p. 173.
[71]M. Planck, Ref. 61, p. 387; R. McCormmach, Ref. 10, p. 40.

DOCTORS AND TEACHERS

We [teachers] do far more damage to our students in other unavoidable ways, and we usually never know it. That is in the nature of our chosen calling, just as doctors kill most of their patients in the long run.

Ian Stewart, in a review of *Mathematical Cranks*, by Underwood Dudley. American Mathematical Monthly **100** (1), 89 (1993).