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Using surface integrals for checking Archimedes' law of buoyancy

F M S Lima

Institute of Physics, University of Brasilia, PO Box 04455, 70919-970 Brasilia-DF, Brazil

E-mail: fabio@fis.unb.br

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Abstract
A mathematical derivation of the force exerted by an *inhomogeneous* (i.e. compressible) fluid on the surface of an *arbitrarily shaped* body immersed in it is not found in the literature, which may be attributed to our trust in Archimedes' law of buoyancy. However, this law, also known as Archimedes' principle (AP), does not yield the force observed when the body is in contact with the container walls, as is more evident in the case of a block immersed in a liquid and in contact with the bottom, in which a *downward* force that *increases with depth* is observed. In this work, by taking into account the surface integral of the pressure force exerted by a fluid over the surface of a body, the general validity of AP is checked. For a body fully surrounded by a fluid, homogeneous or not, a gradient version of the divergence theorem applies, yielding a volume integral that simplifies to an upward force which agrees with the force predicted by AP, as long as the fluid density is a *continuous function of depth*. For the bottom case, this approach yields a downward force that increases with depth, which contrasts to AP but is in agreement with experiments. It also yields a formula for this force which shows that it increases with the area of contact.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The quantitative study of hydrostatic phenomena began in antiquity with Archimedes' treatise *On Floating Bodies—Book I*, where some propositions for the problem of the force exerted by a liquid on a body fully or partially submerged in it are proved [1, 2].[1] In modern texts, his propositions are reduced to a single statement known as Archimedes' law of buoyancy, or simply Archimedes' principle (AP), which states that 'any object immersed in a fluid will experience an upward force equal to the weight of the fluid displaced by the body' [3].[2]

[1] Certainly in connection to the need, at that time of improving shipping by sea routes, of predicting how much additional weight a ship could support without sinking.

[2] Note that Archimedes did not call his discoveries in hydrostatics 'laws', nor did he present them as a consequence of experiments. Instead, he treated them as mathematical theorems, similar to those proposed by Euclid for geometry.

The development of this work had to wait about 18 centuries, until the rise of the scientific method, which was a guide for the experimental investigations in hydrostatics by Stevinus, Galileo, Torricelli, and Pascal, among others[3]. The very long time interval from Archimedes to these experimentalists is a clear indication of how advanced his thoughts were. As pointed out by Netz (based upon a palimpsest discovered recently), Archimedes developed rigorous mathematical proofs for most of his ideas [4]. However, the derivation of the exact force exerted by an *inhomogeneous* fluid on an *arbitrarily shaped* body immersed in it, as will be shown here, demands the knowledge of the divergence theorem, a mathematical tool that was out of reach for the ancients. Therefore, the validity of the Archimedes propositions for this more general case was not formally proved in his original work (see [1]).

By defining the buoyant force (BF) as the net force exerted by a fluid on the portion of the surface of a body (fully or partially submerged) that touches the fluid, the validity of AP in predicting this force can be checked. In fact, the simple case of symmetric solid bodies (e.g., a right-circular cylinder, as found in [5], or a rectangular block, as found in [6]) immersed in a liquid is used in most textbooks for proving the validity of AP. There, symmetry arguments are taken into account to show that the horizontal forces exerted by the liquid cancel and then the net force reduces to the difference of pressure forces exerted on the top and bottom surfaces[4]. The BF is then shown to point upwards, with a magnitude that agrees with AP, which also explains the origin of the BF in terms of an increase of pressure with depth[5]. Note, however, that this proof works only for *symmetric* bodies with horizontal, flat surfaces on the top and the bottom, immersed in an incompressible (i.e. homogeneous) fluid. Although an extension of this result for *arbitrarily shaped* bodies immersed in a *liquid* can be found in some textbooks [7–9], a *formal* generalization for bodies immersed in a *compressible* (i.e. inhomogeneous) fluid is not found in the literature. This certainly induces the readers to believe that it should be very complex mathematically, which is not true, as will be shown here.

The existence of exceptions to Archimedes' law has been observed in some simple experiments in which the force predicted by AP is qualitatively *incorrect* for a body immersed in a fluid and in contact with the container walls. For instance, when a symmetric solid (e.g., a cylinder) is fully submerged in a liquid with a face touching the bottom of a container, a *downward* BF is observed, as long as no liquid seeps under the block [2, 10, 11, 12, 14]. Indeed, the experimental evidence that this force *increases with depth* (see, e.g., [2, 12–14]) clearly contrasts to the *constant* force predicted by AP. These disagreements led some authors to reconsider the completeness or correctness of the AP statement, as well as the definition of BF itself [2, 11–13, 15], which seems to make the things yet more confusing.

On aiming at checking mathematically the validity of AP for arbitrarily shaped bodies immersed in inhomogeneous fluids and intending to elucidate why the bottom case represents an exception to AP, in this work I make use of surface integrals for deriving the exact force exerted by a fluid on the surface of a body immersed in it. When the body does not touch the container walls, a gradient version of the divergence theorem applies, yielding a volume integral that reduces to the weight of the displaced fluid, in agreement with AP. For the bottom case, this approach yields a force that points *downwards* and *increases with depth*, in clear disagreement with AP, but in agreement with experiments. This method also reveals that this

[3] Gravesande also deserves citation, due to his accurate experiment for comparing the force exerted by a liquid on a body immersed in it to the weight of the displaced liquid. This experiment uses a bucket and a metallic cylinder that fits snugly inside the bucket. By suspending the bucket and the cylinder from a balance and bringing it into equilibrium, one immerses the cylinder in a water container. The balance equilibrium is then restored by filling the bucket with water.

[4] Of course, the only physically plausible cause for this upward force is the greater fluid pressure on the bottom of the cylinder in comparison to the pressure on the top.

[5] The origin of the BF is not found in Archimedes' original work [1].

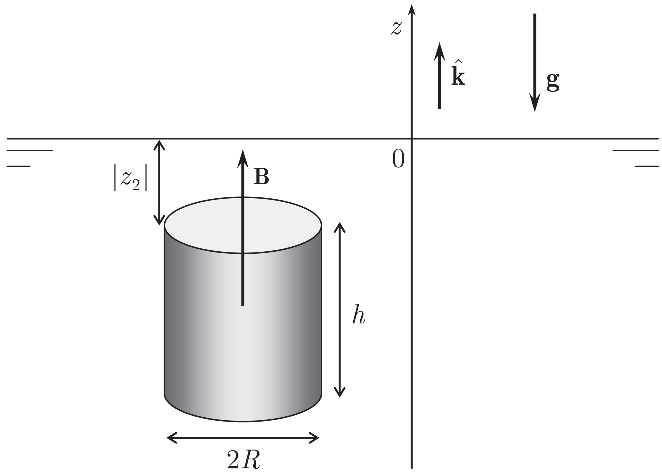

Figure 1. A cylinder of height h and radius R, fully immersed in a liquid. Note that the BF **B** is vertical and directed upwards. The top surface S_2 is kept at a depth z_2 below the liquid–air interface (a plane at $z = 0$).

force *depends on the area of contact* A_b *between the body and the bottom*, being equal to the difference between the product $p_b A_b$ (p_b being the pressure at the bottom) and the weight of the displaced fluid, a result that is not found in the literature and could be explored in undergraduate classes.

2. Buoyant force on a body with arbitrary shape

In modern texts, Archimedes' original propositions are reduced to a single, short statement known as *Archimedes' principle* [3, 6].

> When a body is fully or partially submerged in a fluid, a buoyant force **B** from the surrounding fluid acts on the body. This force is directed upward and has a magnitude equal to the weight $m_f\, g$ of the fluid displaced by the body.

Here, m_f is the mass of the fluid that is displaced by the body and g is the local acceleration of gravity. The BF that follows from AP is simply

$$\mathbf{B} = -\, m_f\, \mathbf{g}\,, \tag{1}$$

where $\mathbf{g} = -g\,\hat{\mathbf{k}}$, $\hat{\mathbf{k}}$ being the unit vector pointing in the z-axis direction, as indicated in figure 1. For an incompressible, homogeneous fluid—i.e. a fluid with a nearly uniform density[6]—, one has $m_f = \rho\, V_f$, and then

$$\mathbf{B} = \rho\, V_f\, g\,\hat{\mathbf{k}}\,, \tag{2}$$

where ρ is the density of the fluid and V_f is the volume of the fluid corresponding to m_f.[7] In most textbooks, some heuristic arguments similar to the Stevinus 'principle of rigidification' (see, e.g., [16]) are taken into account for extending the validity of this formula to *arbitrarily*

[6] For small depths, most liquids behave as practically incompressible fluids. For instance, the density of sea water increases by only 0.5% when the (absolute) pressure increases from 1 to 100 atm (at a depth of 1000 m).

[7] Of course, when the body is fully submerged, V_f equals the volume V of the body.

shaped bodies immersed in a liquid (see, e.g., [3, 5, 6]), without a mathematical justification. The validity of this generalization is checked below.

Our method starts from the basic relation between the pressure gradient in a fluid in equilibrium in a gravitational field $\mathbf{g}$ and its density $\rho = \rho(\mathbf{r})$, with $\mathbf{r}$ being the position vector. From the force balance for an element of volume of the fluid (homogeneous or not) in equilibrium, one finds the well-known *hydrostatic equation* [9, 17]

$$\nabla p = \rho(\mathbf{r})\,\mathbf{g}\,. \tag{3}$$

For a *uniform*, vertical (downward) gravitational field, as will be assumed hereafter[8], this simplifies to

$$\nabla p = \frac{\partial p}{\partial z}\,\hat{\mathbf{k}} = -\rho(z)\,g\,\hat{\mathbf{k}}\,. \tag{4}$$

For a homogeneous (i.e. incompressible) fluid, ρ is a constant (see footnote 6), and then (4) can be readily integrated, yielding

$$p(z) = p_0 - \rho\,g\,z\,, \tag{5}$$

where p_0 is the pressure at $z = 0$, an arbitrary reference level. This linear decrease of pressure with height is known as the *Stevinus law* [3, 5, 6].

Let us now derive a general formula for BF evaluations, i.e. one that works for arbitrarily shaped bodies, fully or partially submerged in a fluid (or set of distinct fluids) homogeneous or not. To avoid confusion, let us *define* the BF as *the net force that a fluid exerts on the part S of the surface* Σ of a body that is effectively in contact with the fluid. Of course, if the body is fully submerged, the surface S coincides with Σ. Consider, then, a body immersed in a fluid in static equilibrium. Since there is no shearing stress in a fluid at rest, the differential element of force d$\mathbf{F}$ it exerts on a differential element of surface dS, at a point P of S in which the fluid touches (and pushes) the body, is normal to S by P. Therefore,

$$\mathrm{d}\mathbf{F} = -p(\mathbf{r})\,\mathrm{d}S\,\hat{\mathbf{n}}\,, \tag{6}$$

where $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r})$ is the *outward* normal unit vector at point P. Note that the pressure force is directed along the *inward* normal to S by P. On assuming that the surface S is piecewise smooth and the vector field $p(\mathbf{r})\,\hat{\mathbf{n}}$ is integrable over S, one finds a general formula for BF evaluations, namely

$$\mathbf{F} = -\iint_S p\,\hat{\mathbf{n}}\,\mathrm{d}S\,. \tag{7}$$

This integral can be easily evaluated for a body with a *symmetric* surface fully submerged in a fluid with *uniform density*. For an arbitrarily shaped body, however, it does not appear to be tractable analytically due to the dependence of the direction of $\hat{\mathbf{n}}$ on the position $\mathbf{r}$ over S, which in turn depends on the (arbitrary) shape of S. However, this task can be easily worked out for a body fully submerged in a homogeneous fluid (ρ = constant) by applying the divergence theorem to the vector field $\mathbf{E} = -p(z)\,\hat{\mathbf{k}}$, which yields a volume integral that evaluates to $\rho\,g\,V\,\hat{\mathbf{k}}$, in agreement with AP, as discussed in some advanced texts [7, 20]. For the more general case of a body fully or partially submerged in an *inhomogeneous* fluid (or a set of fluids), I shall follow a slightly different route here, based upon the following version of the divergence theorem [7, 19].[9]

8 The assumption of a uniform gravitational field is a very good approximation for points near the Earth's surface. However, for problems involving large depths/heights of fluids (e.g., undersea extraction of oil, isostasy in deep crustal layers, and others), the decay of the gravitational field with the distance to the centre of the Earth has to be taken into account. For these applications, a more accurate model (reconsidering equations (4) and (5)) has to be developed.

9 This more general version of the divergence theorem is used here because it is valid for piecewise smooth surfaces, thus covering, e.g., the cases of rectangular blocks and cylinders, in which S is not smooth in *all* its points.

Gradient theorem. *Let R be a bounded region in space whose boundary S is a closed, piecewise smooth surface which is positively oriented by a unit normal vector* $\hat{\mathbf{n}}$ *directed outwards from R. If* $f = f(\mathbf{r})$ *is a scalar function with continuous partial derivatives in all points of an open region that contains R (including S), then*

$$\oiint_S f(\mathbf{r})\, \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_R \nabla f \, \mathrm{d}V .$$

In the appendix, it is shown how the divergence theorem can be used for proving the gradient theorem. The advantage of using this less-known calculus theorem is that it allows us to transform the surface integral in equation (7) into a volume integral of ∇p, a vector that can be easily written in terms of the fluid density via the hydrostatic equation (4). For this, let us substitute $f(\mathbf{r}) = -p(\mathbf{r})$ into both integrals of the gradient theorem. This yields

$$-\oiint_S p(\mathbf{r})\, \hat{\mathbf{n}}\, \mathrm{d}S = -\iiint_{V_f} \nabla p \, \mathrm{d}V , \tag{8}$$

where the surface integral on the left-hand side is, according to our definition, the BF itself whenever the surface S is closed, i.e. when the body is *fully submerged* in a fluid. Let us analyse this more closely.

2.1. A body fully submerged in a fluid

For an arbitrary shape body fully submerged in a fluid (or a set of fluids), by substituting the pressure gradient of equation (4) into equation (8), one finds[10]

$$\mathbf{F} = -\iiint_{V_f} \nabla p \, \mathrm{d}V = \left[\iiint_{V_f} \rho(z) \, \mathrm{d}V \right] g\, \hat{\mathbf{k}} . \tag{9}$$

For the general case of an inhomogeneous, compressible fluid whose density changes with depth, as occurs with gases and high columns of liquids[11], the pressure gradient in equation (9) will be integrable over V_f as long as $\rho(z)$ is a continuous function of depth (in conformity to the hypothesis of the gradient theorem). With this condition, one has

$$\mathbf{F} = \left[\iiint_{V_f} \rho(z) \, \mathrm{d}V \right] g\, \hat{\mathbf{k}} = \left[\iiint_{V} \rho(z) \, \mathrm{d}V \right] g\, \hat{\mathbf{k}} . \tag{10}$$

Since $\iiint_V \rho(z)\, \mathrm{d}V$ is the mass m_f of fluid that would occupy the volume V of the body (fully submerged),

$$\mathbf{F} = m_f\, g\, \hat{\mathbf{k}} , \tag{11}$$

which is an upward force whose magnitude equals the *weight of the fluid displaced by the body*, in agreement with AP as stated in equation (1). This shows that AP remains valid even for an inhomogeneous fluid, as long as the density is a continuous function of depth, a condition fulfilled in most practical situations.

[10] For a homogeneous fluid, one has $\rho(\mathbf{r}) = \rho = \text{const}$, which reduces the integral in equation (9) to $\rho\, g \left(\iiint_{V_f} \mathrm{d}V \right) \hat{\mathbf{k}} = \rho\, g\, V_f\, \hat{\mathbf{k}} = \rho\, g\, V\, \hat{\mathbf{k}}$, which, as expected, agrees with AP.

[11] Being rigorous, all fluids present some increase of density with depth, but the increasing rate for liquids is much smaller than that for gases.

2.2. A body partially submerged in a fluid

The case of an arbitrarily shaped body floating in a fluid with a density $\rho_1(z)$, with its non-submerged part exposed to either vacuum (i.e. a fictitious fluid with null density) or a less dense fluid, is an interesting example of floating in which the exact BF can be compared to the force predicted by AP[12]. This is important for the study of many floating phenomena, from ships in seawater to the isostatic equilibrium of tectonic plates (known in geology as isostasy) [28]. Without loss of generality, let us restrict our analysis to two fluids: one (denser) with density $\rho_1(z)$, we call fluid 1; another (less dense) with density $\rho_2(z) \leqslant \min[\rho_1(z)] = \rho_1(0^-)$, we call fluid 2. For simplicity, I choose the origin $z = 0$ at the planar surface of separation between the fluids, as indicated in figure 2, where the fluid density usually presents a (finite) discontinuity $\rho_1(0^-) - \rho_2(0^+)$. The forces that these fluids exert on the body surface can be evaluated by applying the gradient theorem to each fluid separately, as follows. First, divide the body surface S into two parts: the open surface S_1 below the interface at $z = 0$ and the open surface S_2 above $z = 0$. The integral over the (closed) surface S in equation (7) can then be written as

$$\oiint_S p(z)\, \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S_1} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \iint_{S_2} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S\,, \tag{12}$$

where $\hat{\mathbf{n}}_\mathbf{1}$ ($\hat{\mathbf{n}}_\mathbf{2}$) is the outward unit normal vector at a point of S_1 (S_2), as indicated in figure 2. Let us call S_0 the planar surface, also indicated in figure 2, corresponding to the horizontal cross-section of the body at $z = 0$. By noting that $\hat{\mathbf{n}}_\mathbf{1} = \hat{\mathbf{k}}$ and $\hat{\mathbf{n}}_\mathbf{2} = -\hat{\mathbf{k}}$ in all points of S_0, $p(z)$ being a continuous function, one has

$$\iint_{S_0} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \iint_{S_0} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S = \mathbf{0}.$$

This allows us to use S_0 to generate two closed surfaces, $\widetilde{S}_1$ and $\widetilde{S}_2$, formed by the unions $S_1 \cup S_0$ and $S_2 \cup S_0$, respectively. From equation (12), one has

$$\begin{aligned}\oiint_S p(z)\, \hat{\mathbf{n}}\, \mathrm{d}S &= \iint_{S_1} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \iint_{S_0} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \iint_{S_2} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S + \iint_{S_0} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S \\ &= \oiint_{\widetilde{S}_1} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \oiint_{\widetilde{S}_2} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S\,. \end{aligned} \tag{13}$$

As both $\widetilde{S}_1$ and $\widetilde{S}_2$ are closed surfaces, one can apply the gradient theorem to each of them, separately. This gives

$$\begin{aligned}\mathbf{F} &= -\left(\oiint_{\widetilde{S}_1} p(z)\, \hat{\mathbf{n}}_\mathbf{1}\, \mathrm{d}S + \oiint_{\widetilde{S}_2} p(z)\, \hat{\mathbf{n}}_\mathbf{2}\, \mathrm{d}S \right) \\ &= -\left(\iiint_{V_1} \nabla p\, \mathrm{d}V + \iiint_{V_2} \nabla p\, \mathrm{d}V \right) \\ &= -\left(\iiint_{V_1} \frac{\partial p}{\partial z}\, \mathrm{d}V + \iiint_{V_2} \frac{\partial p}{\partial z}\, \mathrm{d}V \right) \hat{\mathbf{k}}\,, \end{aligned} \tag{14}$$

where V_1 and V_2 are the volumes of the portions of the body below and above the interface at $z = 0$, respectively. From the hydrostatic equation, one has $\partial p/\partial z = -g\, \rho(z)$, which reduces

[12] A null pressure is assumed on the portions of S that are not interacting with any fluid.

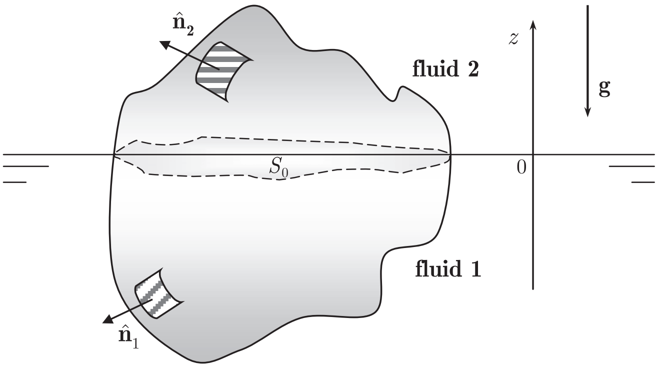

Figure 2. An arbitrarily shaped body floating in a liquid (fluid 1), with the non-submerged part in contact with a less dense, compressible fluid (fluid 2). Note that $\hat{\mathbf{n}}_1$ ($\hat{\mathbf{n}}_2$) is the outward unit vector on the surface S_1 (S_2), as defined in the text. S_0 is the horizontal cross-section at the level of the planar interface between fluids 1 and 2 (at $z = 0$).

the above integrals to

$$\iiint_{V_1} [-\rho_1(z)\, g]\, \mathrm{d}V + \iiint_{V_2} [-\rho_2(z)\, g]\, \mathrm{d}V = -g \left[\iiint_{V_1} \rho_1(z)\, \mathrm{d}V + \iiint_{V_2} \rho_2(z)\, \mathrm{d}V \right]. \tag{15}$$

The latter volume integrals are equivalent to the masses m_1 and m_2 of fluids 1 and 2 displaced by the body, respectively, which reduces the BF to

$$\mathbf{F} = g \left[\iiint_{V_1} \rho_1(z)\, \mathrm{d}V + \iiint_{V_2} \rho_2(z)\, \mathrm{d}V \right] \hat{\mathbf{k}} = (m_1 + m_2)\, g\, \hat{\mathbf{k}}\,. \tag{16}$$

The BF is then upwards and its magnitude is equal to the sum of the weights of the fluids displaced by the body, in agreement with AP in the form given in equation (1). Note that the potential energy minimization technique described in [21–23] cannot provide this confirmation of AP because it works only for rigorously *homogeneous* (i.e. incompressible) fluids. Interestingly, our proof shows that the exact BF can also be found by assuming that the body is fully submerged in a *single* fluid with a variable density $\rho(z)$ that is not continuous, but a *piecewise* continuous function with a (finite) leap discontinuity at $z = 0$.[13]

Although the contribution of fluid 2 to the BF is usually smaller than that of fluid 1, it cannot in general be neglected, as has been done in introductory physics textbooks [24]. In our approach, this corresponds to assuming a constant pressure on all points of the surface S_2 of the emerged portion, which is incorrect. This leads to a null gradient of pressure on the non-submerged part of the body, which erroneously reduces the BF to only

$$\mathbf{F} = \left[\iiint_{V_1} \rho_1(z)\, \mathrm{d}V \right] g\, \hat{\mathbf{k}}\,. \tag{17}$$

Fluid 1 being a liquid, as usual, $\rho_1(z)$ is nearly a constant (let us call it ρ_1), which simplifies this upward force to $\rho_1\, V_1\, g$. This is the result presented for the water–air pair in most textbooks

[13] Interestingly, this suggests that a more general version of the divergence theorem (see the appendix) could be found, in which the requirement of continuity of the components of $\mathbf{E} = \mathbf{E}\,(\mathbf{r})$ is weakened to only a *piecewise continuity*. I have not found such a generalization in the literature.

[3, 5, 6]. It is also the result that can be deduced from Archimedes' original propositions[14], as well as the result found by minimizing the potential energy [21–23]. It is clear that this naive approximate result always *underestimates* the actual BF established in equation (16), being a reasonable approximation only when $m_2 \ll m_1$.[15] The inclusion of the term corresponding to m_2 is then essential for accurate evaluations of the BF, as shown by Lan for a block floating in a liquid [24]. For an arbitrarily shaped body immersed in a liquid–gas fluid system, our equation (15) yields the following expression for the exact BF:

$$\mathbf{F} = \left(\rho_1\, g\, V_1 - \iiint_{V_2} \frac{\partial p}{\partial z}\, \mathrm{d}V \right) \hat{\mathbf{k}}\,. \tag{18}$$

Therefore, the function $p(z)$ on the emerged portion of the body determines the contribution of fluid 2 to the exact BF. If fluid 2 is approximated as an incompressible fluid, i.e. if one assumes $\rho_2(z) \approx \rho_2(0) \equiv \rho_2$, by applying the Stevinus law, one finds

$$\mathbf{F} \approx \left\{ \rho_1\, g\, V_1 - \iiint_{V_2} [-\rho_2(0)\, g]\, \mathrm{d}V \right\} \hat{\mathbf{k}} = (\, \rho_1\, g\, V_1 + \rho_2\, g\, V_2)\, \hat{\mathbf{k}}\,. \tag{19}$$

This is just the result found by Lan by applying AP in the form stated in our equation (2) to fluids 1 and 2, separately, and then summing up the results [24]. Of course, this approximation is better than equation (17), but, contrarily to Lan's opinion, the *correct* result arises only when one takes into account the decrease of ρ_2 with z. This demands the knowledge of a barometric law, i.e. a formula for $p(z)$ with $z > 0$. Fortunately, most known barometric laws are derived just from the hydrostatic equation [17], which allows us to substitute the pressure derivative into equation (18) by $-\rho_2(z)\, g$, finding that

$$\begin{aligned} \mathbf{F} &= \left[\rho_1\, g\, V_1 - \iiint_{V_2} [-\rho_2(z)\, g]\, \mathrm{d}V \right] \hat{\mathbf{k}} \\ &= \left[\rho_1\, V_1 + \iiint_{V_2} \rho_2(z)\, \mathrm{d}V \right] g\, \hat{\mathbf{k}}\,, \end{aligned} \tag{20}$$

which promptly reduces to the exact result found in equation (16).[16]

3. Exceptions to Archimedes' principle

When a body is immersed in a liquid and put in contact with the container walls, as illustrated in figure 3, a BF is observed which does not agree with that predicted by AP. This has been observed in some simple experiments in which a symmetric solid (typically a cylinder or a rectangular block) is fully submerged in a liquid with a face touching the bottom of a container. Let us call this the *bottom case* [10]. In this case, a *downward* BF (according to our definition) is observed if no liquid seeps under the block [10–12, 14]. There is indeed experimental evidence that the magnitude of this force *increases linearly with depth* (see, e.g., [2, 12, 14]), which contrasts with the *constant* force ($= \rho\, V g\, \hat{\mathbf{k}}$) predicted by AP. These experimental results have led some authors to reconsider the completeness or correctness of the AP statement, as well as the definition of BF itself [2, 11–13, 15].

[14] Archimedes certainly knew the existence of air, which was discovered by Empedocles (490–430 BC), but, as pointed out by Rorres, it is doubtful that he was aware of the *buoyancy effects of air*. See note 4 in [25].

[15] In fact, for most applications involving a liquid–air pair, the second term in equation (19) represents only a small correction to the approximation in equation (17) because air density is generally much smaller than typical liquid densities. For instance, at sea level, the air density is only 1.2 kg m^{-3}, whereas the fresh water density is about 1000 kg m^{-3}.

[16] When fluid 2 is vacuum—i.e. the body is *partially submerged* (literally) in fluid 1 only—the exact result in equation (20), above, is indeed capable of furnishing the correct force (namely an upward force with magnitude $\rho_1\, V_1\, g$), which is found by taking the limit as $\rho_2(z)$ tends to zero uniformly.

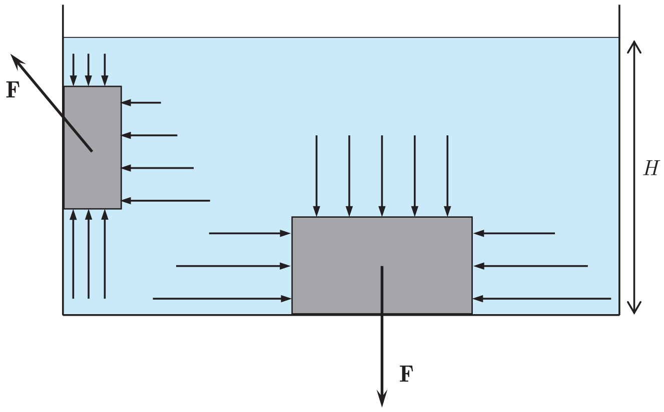

Figure 3. The hydrostatic forces acting on rectangular blocks in contact with the walls of a container. The arrows indicate the pressure forces exerted by the liquid on the surface of each block. The net force exerted by the liquid in each block, i.e. the 'buoyant' force (as defined in the text), is represented by the vector **F**. The larger block represents the *bottom case*.

In spite of the experimental difficulties in studying these exceptions to AP, I shall apply the surface integral approach to determine the exact BF that should be observed in an ideal experiment in which there is no fluid under the block. This will help us to understand why the bottom case represents an exception to AP. For a symmetric solid body (either a cylinder or a rectangular block) with its flat bottom resting on the bottom of a container, as illustrated for the block at the right of figure 3, if no liquid seeps under the block, then the horizontal forces cancel and the *net force exerted by the liquid* reduces to the downward pressure force exerted by the liquid on the top surface [10–12]. From the Stevinus law, $p_{\text{top}} = p_0 + \rho\, g\, |z_{\text{top}}|$, z_{top} being the depth of the top; therefore,

$$\mathbf{F} = -\iint_{S_{\text{top}}} p(z)\, \hat{\mathbf{n}}\, \mathrm{d}S = -p_{\text{top}}\, A\, \hat{\mathbf{k}} = -(p_0 + \rho\, g\, |z_{\text{top}}|) A\, \hat{\mathbf{k}}\,, \qquad (21)$$

where A is the area of the top surface. This *downward* force then *increases linearly with depth*, which clearly contrasts with the force predicted by AP, but agrees with experimental results [2, 12, 14]. In fact, the increase of this force with depth has been the subject of deeper discussions in recent works [2, 12, 13, 26], in which it is suggested that the meaning of the word 'immersed' should be 'fully surrounded by a liquid' instead of 'in contact with a liquid', which would make the 'bottom' case, as well as all other 'contact cases', out of the scope of Archimedes' original propositions, as well as the modern AP statement [2, 12, 26]. Note, however, that this redefinition is deficient because it excludes some common cases of buoyancy such as, for instance, that of a solid (e.g., a piece of cork) floating in a denser liquid (e.g., water). In this simple example, the body is not fully surrounded by a liquid and yet AP works! More recently, other authors have argued that the definition of BF itself should be changed to 'an upward force with a magnitude equal to the weight of the displaced fluid' [13]. However, I have noted that this would make AP a *definition* for the BF and then, logically, AP would not admit any exceptions at all. Faced with the *downward* BF experiments already mentioned, it is clear that this is not a good choice of definition. Therefore, I would like to propose the abandonment of such redefinitions, as they are unnecessary once we admit some exceptions to AP, which is the natural way to treat the *exceptional* cases not realized by Archimedes in his original work.

For a better comparison to the result for arbitrarily shaped bodies that will be derived below, let us write the force found in equation (21) in terms of the pressure p_b at the bottom of the container. As the reader can easily check, this yields

$$\mathbf{F} = -\,(p_b\, A - \rho\, V g)\ \hat{\mathbf{k}}\,. \tag{22}$$

Note that this simple result is for a *'vacuum' contact*, i.e. an ideal contact in which neither liquid nor air is under the block. This is an important point to be taken into account by those interested in developing a downward BF experiment similar to that proposed by Bierman and Kincanon [12], since a part of the bottom of the block with an area A_{air} is intentionally left in contact with air (this comes from their technique to reduce the liquid seepage under the block) [14]. This changes the BF to

$$\mathbf{F} = -\,(p_b\, A - p_0\, A_{\mathrm{air}} - \rho\, V g)\ \hat{\mathbf{k}}\,. \tag{23}$$

For an *arbitrarily shaped body* immersed in a liquid (in the bottom case), let us assume that there is a non-null area A_b of direct contact between the body and the bottom of a container. If no liquid seeps under the block, then the pressure exerted by the *liquid* there at the bottom of the body is of course null. The BF is then

$$\begin{aligned}\mathbf{F} &= -\iint_{S_2\cup A_b} p(z)\hat{\mathbf{n}}\,\mathrm{d}S = -\left[\iint_{S_2} p(z)\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{A_b} 0(-\hat{\mathbf{k}})\,\mathrm{d}S\right]\\ &= -\iint_{S_2} p(z)\hat{\mathbf{n}}\,\mathrm{d}S.\end{aligned}$$

In view of applying the gradient theorem, one needs a surface integral over a *closed* surface. By creating a fictitious closed surface $\Sigma = S_2 \cup A_b$ on which the pressure forces will be exerted as if the body would be fully surrounded by the liquid (i.e. one assumes a constant pressure p_b over the horizontal surface A_b), one has

$$\begin{aligned}\mathbf{F} &= -\iint_{S_2} p(z)\,\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{A_b} p_b\,(-\hat{\mathbf{k}})\,\mathrm{d}S + \iint_{A_b} p_b\,(-\hat{\mathbf{k}})\,\mathrm{d}S\\ &= -\left[\oiint_{\Sigma} p(z)\,\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{A_b} p_b\,(-\hat{\mathbf{k}})\,\mathrm{d}S\right] = -\oiint_{\Sigma} p(z)\,\hat{\mathbf{n}}\,\mathrm{d}S - p_b\,\hat{\mathbf{k}}\iint_{A_b}\mathrm{d}S\\ &= \rho\,V\,g\,\hat{\mathbf{k}} - p_b\,A_b\,\hat{\mathbf{k}} = -(p_b\,A_b - \rho\,V g)\,\hat{\mathbf{k}}\,.\end{aligned} \tag{24}$$

This is again a *downward* BF that increases linearly with depth, since $p_b = p_0 + \rho\, g\, H$, H being the height of the liquid column above the bottom, as indicated in figure 3. This result for arbitrarily shaped bodies is not found in the literature. Incidentally, this result suggests that the only exceptions to AP, for fluids in equilibrium, are those cases in which ∇p is not a piecewise continuous function over the whole surface Σ of the body; otherwise the results of the previous section guarantee that the BF points upwards and has a magnitude $\rho\, V g$, in agreement with AP. This includes all contact cases, since the boundary of the contact region is composed of points around which the pressure leaps (i.e. changes discontinuously) from a strictly positive value $p(z)$ (on the liquid side) to a smaller (ideally null) pressure (at the contact surface).

Therefore, in these points, the pressure is not a differentiable function (because it is not even a continuous function), which impedes the application of the gradient theorem[17].

4. Conclusions

Here in this paper, I have drawn the attention of the readers to the fact that the BF predicted by AP can be derived mathematically even for bodies of arbitrary shape, fully or partially submerged in a fluid, homogeneous or not, based only upon the validity of the hydrostatic equation and the gradient theorem. For that, I first define the BF as the net force exerted by a fluid on the portion of the surface of the body that is pressed by the fluid. Then, the exact BF becomes a surface integral of the pressure force exerted by the fluid, which can be easily evaluated when the body is fully surrounded by the fluid. In this case, the gradient theorem allows one to convert that surface integral into a volume integral which promptly reduces to an upward force with a magnitude equal to the weight of the displaced fluid (as predicted by AP), as long as the fluid density is a *continuous function of depth*.

Finally, some cases were pointed out in which AP fails and this could help students (even teachers) to avoid erroneous applications of this physical law. The exact force in one of these exceptional cases is determined here by applying our surface integral approach to a body immersed in a liquid and in contact with the bottom of a container. In this case, our result agrees with some recent experiments in which it is shown that the force exerted by the liquid is a *downward* force that *increases linearly with depth*, in clear contrast to the force predicted by AP. The method introduced here is indeed capable of providing a formula for this downward force, valid for bodies with arbitrary shapes, which involves the area of contact. Since Archimedes was one of the greatest geniuses of the ancient world, it would not be surprising that he had enunciated his theorems with remarkable precision and insight; however, there are some instances he did not realize. These cases are shown here to be exceptions to AP; thus, it would not be sensible to make a great effort to keep AP valid without exceptions at the cost of deficient redefinitions.

Since the method presented here is not so complex mathematically, involving only some basic rules of vector calculus, it could be included or mentioned in textbooks, at least in the form of a reference that could be looked up by the more interested readers.

Appendix. Proof of the gradient theorem

Let us show how Gauss's divergence theorem (see, e.g., [7, 8, 18, 19]) can be applied to prove the gradient theorem.

[17] Another kind of exception to AP is the 'balloon' case, investigated by Sharma in a series of accurate experiments [27]. It was observed that the volume of the immersed part of a balloon floating on water, with its non-submerged part in contact with air, deviates from that predicted by AP, the deviations being dependent on the form of the balloon. Again, as the body surface is fully surrounded by two fluids (water and air), if all conditions for applicability of the gradient theorem were satisfied, our results in the previous section would yield the same BF as predicted by AP. Therefore, the only possibility for explaining this exception to AP is the presence of a non-negligible *surface tension* on the region of contact between the balloon sheath and the water, which creates an additional upthrust that makes the balloon float, displacing less water than the amount predicted by AP. Therefore, the argument presented by Sharma that AP should be valid only for certain shapes must be incorrect.

Divergence theorem. *Suppose that R and S satisfy the conditions mentioned in the gradient theorem. If* $\mathbf{E} = \mathbf{E}(\mathbf{r})$ *is a vector field whose components have continuous partial derivatives in all points of V (including S), then*

$$\oiint_S \mathbf{E} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_R \nabla \cdot \mathbf{E} \, \mathrm{d}V .$$

Proof (of gradient theorem). Let P be a point over the closed surface S that bounds R. Suppose that $f = f(\mathbf{r})$ has continuous partial derivatives at every point in R, including those at S. By choosing $\mathbf{E} = f\,\mathbf{c}$, with $\mathbf{c} \neq \mathbf{0}$ being an arbitrary constant vector, and substituting it into the integrals of the divergence theorem, above, one finds

$$\oiint_S (f\,\mathbf{c}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_R \nabla \cdot (f\,\mathbf{c}) \, \mathrm{d}V . \tag{A.1}$$

Since $\nabla \cdot \mathbf{c} = 0$, $\nabla \cdot (f\,\mathbf{c}) = f\,(\nabla \cdot \mathbf{c}) + \mathbf{c} \cdot \nabla f = \mathbf{c} \cdot \nabla f$. Therefore,

$$\oiint_S \mathbf{c} \cdot (f \hat{\mathbf{n}}) \, \mathrm{d}S = \iiint_R \mathbf{c} \cdot (\nabla f) \, \mathrm{d}V ,$$

which implies that

$$\mathbf{c} \cdot \left(\oiint_S f \hat{\mathbf{n}} \, \mathrm{d}S - \iiint_R \nabla f \mathrm{d}V \right) = 0.$$

By hypothesis, $\mathbf{c} \neq \mathbf{0}$. If the vector in parentheses, above, were not null, it would always be perpendicular to $\mathbf{c}$, in order to nullify the scalar product, which is impossible because $\mathbf{c}$ is an arbitrary vector. Therefore, one has to conclude that

$$\oiint_S f \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_R \nabla f \mathrm{d}V .$$

□

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