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Brachistochrones with loose ends

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Abstract

The classical problem of the brachistochrone asks for the curve down which a body sliding from rest and accelerated by gravity will slip (without friction) from one point to another in least time. In undergraduate courses on classical mechanics, the solution of this problem is the primary example of the power of variational calculus. Here, we address the generalized brachistochrone problem that asks for the fastest sliding curve between a point and a given curve or between two given curves. The generalized problem can be solved by considering variations with varying end points. We will contrast the formal solution with a much simpler solution based on symmetry and kinematic reasoning. Our exposition should encourage teachers to include variational problems with free boundary conditions in their courses and students to try simple, intuitive solutions first.

1. Introduction

The Brachistochrone Transit Company (BTC) [1], well known for its high-speed, gravity-propelled point-to-point transportation services, wanted to enter the coal chute market. Coal chutes are usually used to overcome a given horizontal distance Δx, the vertical distance Δy being irrelevant. The chief engineer of BTC, Dr L A Grange, faced the problem of designing the optimal chute. Dr Grange immediately realized that the optimal coal chute is given by one of the most successful sliding devices of the company, the CYCLOID™. His argument is as follows:

(A) The optimal coal chute will run between two points (x, y) and $(x + \Delta x, y + \Delta y)$. But the fastest sliding curve between two given points is a cycloid.

The problem then is to select the *best* cycloid or, equivalently, the corresponding vertical distance Δy.

Coal chutes that are used in practice are simple inclined planes. Under the usual assumptions (no friction, start from rest, homogeneous gravitational acceleration g), the time to cover a horizontal distance Δx by sliding down a straight line with inclination α reads

$$T_s(\alpha) = \sqrt{\frac{\Delta x}{g}}\sqrt{\frac{2}{\sin\alpha\cos\alpha}}. \tag{1}$$

This time is minimal for $\alpha = \frac{\pi}{4}$, i.e., for a line that spans the same horizontal and vertical distances. The minimal time is

$$T_s^\star = T_s\left(\frac{\pi}{4}\right) = 2\sqrt{\frac{\Delta x}{g}}. \tag{2}$$

A (downward) cycloid with starting point (0, 0) is parameterized by

$$x(t) = a(t - \sin t) \qquad y(t) = -a(1 - \cos t) \qquad 0 \leqslant t \leqslant t_0, \tag{3}$$

where the constants a and t_0 are determined by the end point $(\Delta x, \Delta y)$. The time to slide down a cycloid to $(\Delta x, \Delta x)$ is

$$T_c = \frac{t_0}{\sqrt{1-\cos t_0}}\sqrt{\frac{\Delta x}{g}} = 1.826\sqrt{\frac{\Delta x}{g}} \tag{4}$$

where $t_0 = 2.412\ldots$ is the positive root of $t - \sin t = 1 - \cos t$. This is faster than along the straight line, but is it the fastest way? Dr Grange said no, and he gave the following argument:

(B) Suppose we are sliding down the optimal curve. When we are very close to the vertical line at Δx, we can assume that we will reach it in the next infinitesimal time step $\mathrm{d}t$. For this infinitesimal time step, we can ignore the acceleration and assume that we will continue with constant velocity. But the fastest way to the vertical line is to move horizontally, i.e., to *meet the target curve under a right angle.*

Dr Grange quickly calculated that the cycloid that intersects the vertical line at Δx under a right angle is that with end point $\left(\Delta x, \frac{2}{\pi}\Delta x\right)$ (figure 1), and the time to slide down this cycloid reads

$$T_c^\star = \sqrt{\pi}\sqrt{\frac{\Delta x}{g}} = 1.772\sqrt{\frac{\Delta x}{g}}. \tag{5}$$

This is in fact the optimum as Dr Grange verified by trying nearby cycloids. Dr Grange reported the solution to his boss, and BTC became the major player in the coal chute industry.

2. Variations with varying ends

The story of Dr Grange and BTC illustrates two facts: (1) the problem of the brachistochrone with varying end point(s) is largely unknown and (2) its solution requires only a simple argument in addition to the solution of the classical, fixed end point version. In his 1696 paper [2], Johann Bernoulli challenged the learned world with the fixed end point problem (English translation from [3]):

> Let two points A and B given in a vertical plane. To find the curve that point M, moving on a path AMB, must follow that, starting from A, it reaches B in the shortest time under its own gravity.

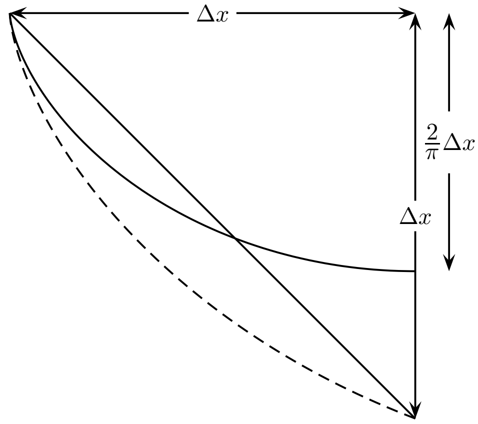

Figure 1. Covering a horizontal distance Δx by sliding down the fastest straight line, a cycloid with the same end point (dashed) or the optimal cycloid (solid).

It is this version that dominates modern textbooks. The more general version of the problem, where A or B (or both) are free to move along given curves, is rarely discussed. Notable exceptions are Gelfand and Fomin [4] or Weinstock [5]. This is astonishing considering the fact that the generalized problem was already solved by Joseph Louis Lagrange in 1760 [6], in the same paper that laid the foundation of variational calculus. Before we present the formal solution let us reconsider Dr Grange's arguments. Obviously, his argument (A) is generally true, i.e., the brachistochrone is always found among the solutions of the corresponding Euler differential equation. The end points (fixed or free) are only used to fix the free parameters of the solution. His argument (B) is also much more general than suggested by the coal chute example. It is a kinematic argument that applies for *any* gravitational field and *any* target curve, not just the vertical line: a brachistochrone will intersect its target curve orthogonally. Let us see whether this is confirmed by the variational calculus.

We want to solve the following variational problem: find the minimum of

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x \tag{6}$$

with respect to variations of y and variations of the end points (x_0, y_0), (x_1, y_1). To find the extremum of the functional $J[y]$, we consider a family of functions $y(x, \eta)$ such that $y(x, 0) = y(x)$. The variations of y and y' are defined as usual by

$$\delta y = \left.\frac{\partial y(x,\eta)}{\partial \eta}\right|_{\eta=0} \cdot \eta$$

and

$$\delta y' = \frac{\partial}{\partial \eta}\left[\frac{\partial y(x,\eta)}{\partial x}\right]\Bigg|_{\eta=0} \cdot \eta = \frac{\mathrm{d}}{\mathrm{d}x}\delta y.$$

We also vary the end points $(x_i(\eta), y_i(\eta))$ such that $(x_i(0), y_i(0)) = (x_i, y_i)$ for $i = 0, 1$. We only need to consider variations of the x-coordinates,

$$\delta x_i = \left.\frac{\mathrm{d}x_i(\eta)}{\mathrm{d}\eta}\right|_{\eta=0} \cdot \eta,$$

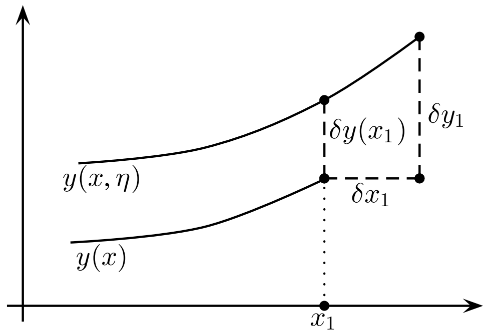

Figure 2. Variation of a curve $y(x)$ and its end point.

because the variations of the y-coordinates are given by δx_i and the variation of the curve $y(x)$ (figure 2):

$$\delta y_i = \delta y(x_i) + y'(x_i)\delta x_i. \tag{7}$$

For this family of functions and end points, $J[y]$ becomes a function of η,

$$J(\eta) = \int_{x_0(\eta)}^{x_1(\eta)} F(x, y(x,\eta), y'(x,\eta))\,\mathrm{d}x. \tag{8}$$

Differentiating $J(\eta)$ at $\eta = 0$ and multiplying by η yields the first variation of J:

$$\delta J = [F\delta x]_0^1 + \int_{x_0}^{x_1} (F_y\delta y + F_{y'}\delta y')\,\mathrm{d}x, \tag{9}$$

where we have used the shorthand $F_y = \partial F/\partial y$, etc and

$$[F\delta x]_0^1 = F(x_1, y(x_1), y'(x_1))\delta x_1 - F(x_0, y(x_0), y'(x_0))\delta x_0.$$

Partial integration of the second term in the integral in (9) yields

$$\int_{x_0}^{x_1} F_{y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\,\mathrm{d}x = [F_{y'}\delta y]_{x_0}^{x_1} - \int_{x_0}^{x_1}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right)\delta y\,\mathrm{d}x.$$

This and (7) allow us to write the variation of J as

$$\delta J = [(F - y'F_{y'})\delta x + F_{y'}\delta y]_0^1 + \int_{x_0}^{x_1}\left(F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right)\delta y\,\mathrm{d}x. \tag{10}$$

A minimum of J implies $\delta J = 0$. In particular, $\delta J = 0$ for the subset of variations which leave the end points fixed, i.e., for $\delta x_i = 0$ and $\delta y_i = 0$. This implies that the integrand in (10) must vanish, which brings us to the Euler equation

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'} = 0. \tag{11}$$

This proves Dr Grange's claim (A) that the optimal curves for varying end points are from the same family of curves that are optimal for fixed end points. If we allow varying end points as well, $\delta J = 0$ in addition implies

$$(F - y'F_{y'})|_{x=x_i}\delta x_i + F_{y'}|_{x=x_i}\delta y_i = 0 \qquad i = 0, 1. \tag{12}$$

Now we consider variations where the end points lie on given curves $g_0(x_0, y_0) = 0$ and $g_1(x_1, y_1) = 0$. This gives us another equation for δx_i and δy_i, namely,

$$\frac{\partial g_i(x_i, y_i)}{\partial x}\delta x_i + \frac{\partial g_i(x_i, y_i)}{\partial y}\delta y_i = 0 \qquad i = 0, 1. \tag{13}$$

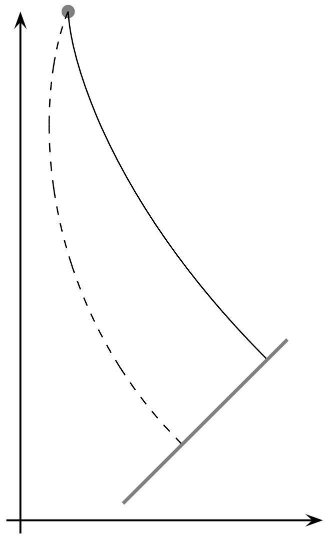

Figure 3. Brachistochrones for the uniform gravitational field (solid) and $1/r$-potential (dashed) connecting a point with a straight line (grey). Both brachistochrones intersect the target curve orthogonally. The target curve is $y = x$ and the starting point is $(1, 2)$ The potentials are $v(x, y) = \sqrt{y_0 - y}$ (uniform field) and $\sqrt{r^{-1}(x, y) - r^{-1}(x_0, y_0)}$ with $r(x, y) = \sqrt{x^2 + y^2}$. For the numerical solution in the $1/r$-potential we used the integral representation of [7].

For a non-trivial solution to exist, the determinant of the coefficients of the system (12) and (13) has to vanish, i.e., we require

$$\partial_x g_i F_{y'}|_{x=x_i} = \partial_y g_i (F - y' F_{y'})|_{x=x_i}. \tag{14}$$

This equation is known as the transversality condition [4]. The curve $y(x)$ satisfying (14) is said to be transversal to the curves $g_i(x, y) = 0$.

Let us now consider variational problems of the brachistochrone type, i.e.,

$$J[y] = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{v(x, y)}\, \mathrm{d}x, \tag{15}$$

where $v(x, y)$ denotes the velocity of the particle at position (x, y). For problems like this, we obtain

$$F_{y'} = \frac{y'}{1 + y'^2} F \qquad F - y' F_{y'} = \frac{1}{1 + y'^2} F,$$

and the transversality conditions simplify to

$$\partial_x g_i(x_i, y_i) y'(x_i) = \partial_y g_i(x_i, y_i). \tag{16}$$

The meaning of this equation becomes more transparent if we assume that the curve $g_i(x, y) = 0$ has an explicit representation $y = \phi_i(x)$. Then (16) can be written as the orthogonality relation

$$\phi_i'(x_i) y'(x_i) = -1.$$

Hence for functionals of the form (15), transversality reduces to orthogonality. This proves Dr Grange's claim (B) for arbitrary gravitational potentials (coded into $v(x, y)$) and arbitrary target curves. Figure 3 shows brachistochrones for the uniform field and the $1/r$-potential running between a point and a straight line. The orthogonal intersection is clearly visible.

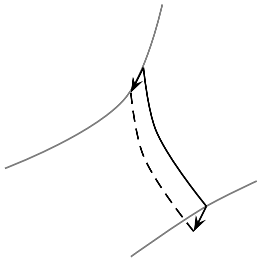

Figure 4. An infinitesimal translation of a sliding curve connecting two curves can save time if the slopes of the curves in the connected points are different.

Note that orthogonality also applies for a varying start point. For $v(x, y) = \text{const}$, this means that the shortest *distance* between two curves is given by a straight line that intersects both curves orthogonally. Note also that the same argument (formal and Dr Grange style) applies in higher dimensions: the brachistochrone that connects a point with a surface is orthogonal to the surface.

3. Visiting Dr Grange

When we showed our analysis to Dr Grange, he was pleased that we confirmed his heuristic reasoning. He pointed out, however, that our treatment of the varying starting point does not properly reflect the situation in the problem of the brachistochrone. Here the initial velocity is usually zero, hence the integrand F itself depends on the starting point (x_0, y_0) via

$$F(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y_0 - y)}}.$$

So, what is the brachistochrone that connects two given curves when we start from rest? Dr Grange remarked that his arguments (A) and (B) are still valid, hence the type of curve (cycloid) is fixed and the orthogonality condition at the end point holds. For the starting point Dr Grange gave the following argument:

(C) Consider a sliding curve that connects two points. If we shift the whole curve by a constant vector, the time to slide down the shifted curve is the same as before. This follows from the translational symmetry of the uniform gravitational field. Let the sliding curve connect two curves $\mathcal{C}_0$ and $\mathcal{C}_1$. We move the whole sliding curve a little bit parallel or antiparallel to the tangent of $\mathcal{C}_0$ at the starting point. If the slope of $\mathcal{C}_1$ at the end point is different from the slope of the tangent, we can arrange that the shifted curve will intersect $\mathcal{C}_1$ before it reaches its end point (figure 4). Hence we could save time by sliding down the shifted curve and, consequently, *a brachistochrone that connects two curves intersects both curves at points of equal slope.*

Impressed by his intuition, we challenged Dr Grange by asking for the brachistochrone with varying starting point but fixed end point. Dr Grange smiled and gave the following argument:

(D) Let us again move the whole brachistochrone a little bit tangentially to the starting curve. The end point will move too, and if we imagine that its movement traces the tangent of a target curve with the same slope as the tangent of the starting curve, we know from (C) that our brachistochrone is also the brachistochrone for this curve-to-curve problem. From (B) we know that the brachistochrone is orthogonal to the imaginary target curve.

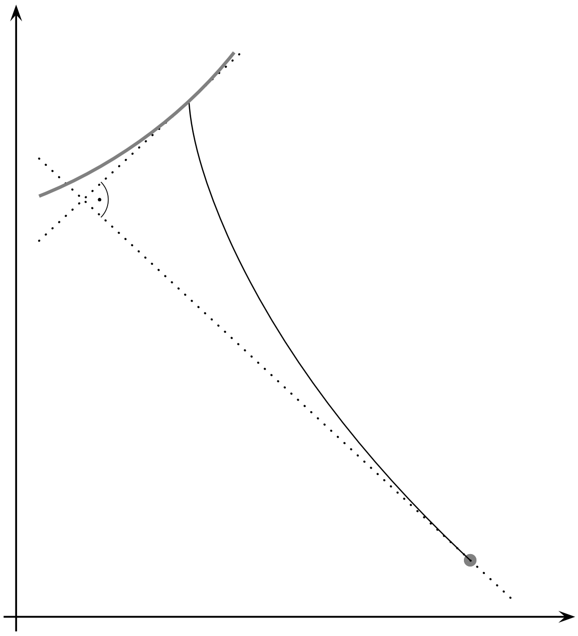

Figure 5. Brachistochrone in a uniform gravitational field from a curve to a point. The slope of the brachistochrone at the target point is orthogonal to the slope of the curve at the starting point.

Hence (B) and (C) together imply that *the slope of the brachistochrone at the fixed end point is orthogonal to the slope of the starting curve at the starting point (figure 5).*

We thanked Dr Grange and headed back to our lab to prove his claims.

4. Brachistochrones connecting curves

We consider functionals where the integrand depends explicitly on the starting point (like in the brachistochrone problem with zero initial velocity):

$$J[y] = \int_{x_0}^{x_1} F(x, y, y'; x_0, y_0)\,\mathrm{d}x. \tag{17}$$

Repeating the arguments from section 2 we obtain

$$\delta J = [(F - y'F_{y'})\delta x + F_{y'}\delta y]_0^1 + \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right)\delta y\,\mathrm{d}x + \int_{x_0}^{x_1} (F_{x_0}\delta x_0 + F_{y_0}\delta y_0)\,\mathrm{d}x. \tag{18}$$

Again we see that $\delta J = 0$ implies the Euler equation (11). The transversality conditions change, however, due to the extra integral. For the rest of this section, we will assume that F depends on x_0 and y_0 only through $F(x - x_0, y - y_0, y')$ as for the brachistochrone in a uniform gravitational field. Then we can evaluate the extra integrals using the Euler equation (11):

$$\int_{x_0}^{x_1} F_{y_0}\,\mathrm{d}x = -\int_{x_0}^{x_1} F_y\,\mathrm{d}x = -\int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\,\mathrm{d}x = -[F_{y'}]_{x_0}^{x_1}. \tag{19}$$

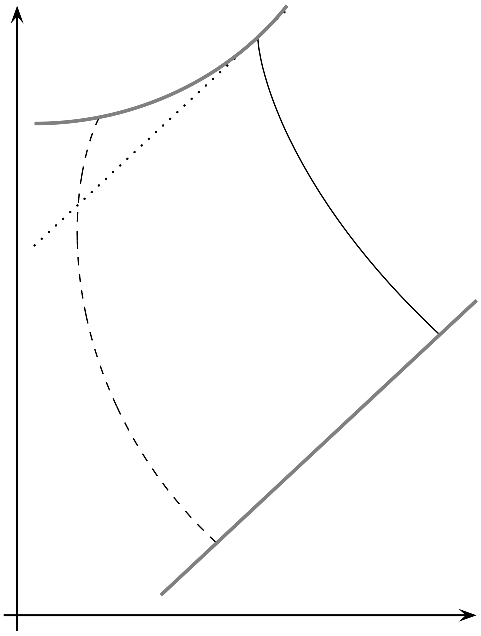

Figure 6. Brachistochrones for the uniform gravitational field (solid) and $1/r$-potential (dashed) connecting a circle with a straight line (grey). Both brachistochrones intersect the target curve orthogonally, but only the brachistochrone in a uniform field connects points of equal slope. The target curve is $y = x$ and the circle is $(x-1)^2 + (y-5/2)^2 = 1/4$. The potentials are the same as in figure 3.

To evaluate the second integral, we use the Euler equation (11) and

$$\frac{\mathrm{d}}{\mathrm{d}x}F = F_x + F_y y' + F_{y'} y''. \tag{20}$$

We find

$$\begin{aligned}\int_{x_0}^{x_1} F_{x_0}\,\mathrm{d}x &= -\int_{x_0}^{x_1} F_x\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} (F_y y' + F_{y'} y'')\,\mathrm{d}x - [F]_{x_0}^{x_1} \\ &= \int_{x_0}^{x_1} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right) y' + F_{y'} y''\right] \mathrm{d}x - [F]_{x_0}^{x_1}.\end{aligned}$$

Partial integration provides us with

$$\int_{x_0}^{x_1} F_{x_0}\,\mathrm{d}x = [F_{y'} y' - F]_{x_0}^{x_1}. \tag{21}$$

If we plug (19) and (21) into (18), we finally obtain

$$\begin{aligned}\delta J = \int_{x_0}^{x_1} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\right)\delta y\,\mathrm{d}x + F_{y'}(x_1, y_1, y_1')(\delta y_1 - \delta y_0) \\ + [F(x_1, y_1, y_1') - y'(x_1)F_{y'}(x_1, y_1, y_1')](\delta x_1 - \delta x_0).\end{aligned} \tag{22}$$

Now $\delta J = 0$ implies $\delta y_1 = \delta y_0$ for $\delta x_1 = \delta x_0$, which proves Dr Grange's claim (C) that a brachistochrone between two curves meets both curves on points of equal slope. If we set

$\delta x_1 = \delta y_1 = 0$, we have the problem of the brachistochrone from a curve to a fixed end point. In this case, the transversality conditions reduce to

$$0 = F_{y'}(x_1, y_1, y_1')\delta y_0 + [F(x_1, y_1, y_1') - y'(x_1)F_{y'}(x_1, y_1, y_1')]\delta x_0. \tag{23}$$

Comparison with (12) for $i = 0$ proves Dr Grange's claim (D).

Note that the 'equal slope' argument (C) relies on the translational invariance of the uniform gravitational field (which we used in our proof by assuming $F = F(x-x_0, y-y_0, y')$). It does not hold in potentials like the $1/r$-potential (figure 6). The same is true for claim (D), although the 'orthogonality' argument (B) applies for any potential.

5. Conclusions

We have seen that the generalized brachistochrone problem that allows the end points to vary along given curves is not much different from the classical version with fixed end points. In all cases, the brachistochrone is given by the solution of the same Euler equation. The algebraic equations that fix the free parameters in this solution depend on the boundary conditions, however. For brachistochrones from a point to a curve, we have the orthogonality condition, i.e., the brachistochrone intersects the target curve orthogonally. Brachistochrones from a curve to a given point are determined by the condition that the slope of the curve at the starting point is orthogonal to the slope of the brachistochrone at the end point. Finally, a brachistochrone from curve to curve is determined by the orthogonality condition and the additional requirement that it connects two points of equal slope. The two latter conditions only hold for homogeneous gravitational potentials, whereas the orthogonality condition holds for arbitrary potentials.

The fact that these results can be derived by simple, intuitive arguments should encourage teachers to include the free boundary version when they discuss the brachistochrone problem. The intuitive solution of the free boundary problem helps to understand what a variation actually means, a concept that many students do not easily grasp. Working out the formal solution on the other hand is a good exercise in using the machinery of variational calculus.

Acknowledgment

We are grateful to Dr L A Grange for sharing his insights with us.

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